Quantization of Constrained Systems*

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Abstract

The present article is primarily a review of the projection-operator approach to quantize systems with constraints. We study the quantization of systems with general first- and second-class constraints from the point of view of coherent-state, phase-space path integration, and show that all such cases may be treated, within the original classical phase space, by using suitable path-integral measures for the Lagrange multipliers which ensure that the quantum system satisfies the appropriate quantum constraint conditions. Unlike conventional methods, our procedures involve no δ-functionals of the classical constraints, no need for dynamical gauge fixing of first-class constraints nor any average thereover, no need to eliminate second-class constraints, no potentially ambiguous determinants, as well as no need to add auxiliary dynamical variables expanding the phase space beyond its original classical formulation, including no ghosts. Besides several pedagogical examples, we also study: (i) the quantization procedure for reparameterization invariant models, (ii) systems for which the original set of Lagrange mutipliers are elevated to the status of dynamical variables and used to define an extended dynamical system which is completed with the addition of suitable conjugates and new sets of constraints and their associated Lagrange multipliers, (iii) special examples of alternative but equivalent formulations of given first-class constraints, as well as (iv) a comparison of both regular and irregular constraints.

*To appear in the proceedings of the 39th Schladming Winter School on “Methods of Quantization”, February 26-March 4, 2000, Schladming, Austria.

1 INTRODUCTION

1.1 Initial comments

The quantization of systems with constraints is important conceptually as well as practically. Principal techniques for the quantization of such systems involve conventional operator techniques [1], path integral techniques in terms of the original phase space variables [2], extended operator techniques involving ghost variables in addition to the original variables and extended path integral techniques also including ghost fields (see, e.g., [3, 4, 5]). However, these standard approaches are generally not unambiguous and may exhibit certain difficulties in application. A recent review [6] carefully analyzes these traditional methods and details their weaknesses as well as their strengths.

Canonical quantization generally requires the use of Cartesian coordinates and not more general coordinates [7]. Therefore, whenever we consider a dynamical system without any constraints whatsoever, we assume that the phase space of the unconstrained system is flat and admits a standard quantization of its canonical variables either in an operator form or in an equivalent path integral form. Next, suppose constraints exist, which, for the sake of discussion, we choose as a closed set of first-class constraints; extensions to treat more general constraints are presented in later sections. Whenever there are constraints the original set of variables is no longer composed solely of physical variables but now contains some unphysical variables as well. While such variables cause little concern from a classical standpoint, they are viewed as highly unwelcome from a quantum standpoint inasmuch as one generally wants to quantize only physical variables. Thus it is often deemed necessary to eliminate the unphysical variables leaving only the true physical degrees of freedom. Quantization of the true degrees of freedom is supposed to proceed as in the initial step. In the general case, however, a quantization of the remaining degrees of freedom is not straightforward or perhaps not even possible because the physical (reduced) phase space is non-Euclidean meaning that an obstruction has arisen where none existed before. An obstruction generally precludes the existence of self-adjoint (observable!) canonical operators satisfying the canonical commutation relations. In path integral treatments, such obstructions arise from the introduction of delta functionals that enforce the classical constraints and the concomitant need

to introduce subsidiary delta functionals to select a compatible dynamical gauge in order to introduce a canonical sympletic structure on the physical phase space that generally is not flat. These are fundamental problems that seem difficult to overcome.

This article reviews a middle ground in the quantization procedure of systems with constraints which may be called the *projection-operator, coherent-state approach*. Briefly stated, quantization of the original, unconstrained variables proceeds without obstruction or ambiguity, while constraints are enforced by means of a well-chosen projection operator projecting the original Hilbert space onto the physical Hilbert subspace. This conservative framework is presented in the form of a phase-space path integral with the help of coherent states (which, while convenient, are not necessary). The difference between the present approach and other functional integral methods may be attributed to an alternative choice for the integration measure for the Lagrange multiplier variables. The present approach may be traced from [8]. In addition, some aspects of the projection operator approach have been presented in unpublished work of Shabanov [9]; see also [10].

1.2 Classical backround

For our initial discussion, let us briefly review the classical theory of constraints. Let $\{p_j, q^j\}$, $1 \leq j \leq J$, denote a set of dynamical variables, $\{\lambda^a\}$, $1 \leq a \leq A$, a set of Lagrange multipliers, and $\{\phi_a(p,q)\}$ a set of constraints. Then the dynamics of a constrained system may be summarized in the form of an action principle by means of the classical action (summation implied)

$$I = \textstyle\int [p_j \dot{q}^j - H(p,q) - \lambda^a \phi_a(p,q)]\, dt\ . \tag{1}$$

The resultant equations that arise from the action read

$$\begin{aligned}
\dot{q}^j &= \frac{\partial H(p,q)}{\partial p_j} + \lambda^a \frac{\partial \phi_a(p,q)}{\partial p_j} \equiv \{q^j, H\} + \lambda^a \{q^j, \phi_a\}\ , \\
\dot{p}_j &= -\frac{\partial H(p,q)}{\partial q^j} - \lambda^a \frac{\partial \phi_a(p,q)}{\partial q^j} \equiv \{p_j, H\} + \lambda^a \{p_j, \phi_a\}\ , \\
\phi_a(p,q) &= 0\ ,
\end{aligned} \tag{2}$$

where $\{\cdot,\cdot\}$ denotes the Poisson bracket. The set of conditions $\{\phi_a(p,q) = 0\}$ defines the *constraint hypersurface*. If the constraints satisfy

$$\{\phi_a(p,q), \phi_b(p,q)\} = c_{ab}{}^{c}\, \phi_c(p,q)\ , \tag{3}$$

$$\{\phi_a(p,q), H(p,q)\} = h_a{}^b\,\phi_b(p,q)\ , \tag{4}$$

then we are dealing with a system of first-class constraints. If the coefficients $c_{ab}{}^c$ and $h_a{}^b$ are constants, then it is a closed system of first-class constraints; if they are suitable functions of the variables p, q, then it is called an open first-class constraint system. If (3) fails, or (3) and (4) fail, then the constraints are said to be second class (see below).

For first-class constraints it is sufficient to impose the constraints at the initial time inasmuch as the equations of motion will ensure that the constraints are fulfilled at all future times. Such an initial imposition of the constraints is called an *initial value equation*. Furthermore, the Lagrange multipliers are not determined by the equations of motion; rather the solutions of the equations of motion depend on them. By specifying the Lagrange multipliers, the solution can be forced to satisfy an additional ("gauge") condition. Observable quantities are gauge invariant and, hence, do not depend on the gauge abritrariness. For second-class constraints, on the other hand, the Lagrange multipliers are determined by the equations of motion in such a way that the constraints are satisfied for all time.

In the remainder of this section we review standard quantization procedures for systems with closed first-class constraints, both of the operator and path integral variety, pointing out some problems in each approach.

1.3 Quantization first: Standard operator quantization

For a system of closed first-class constraints we assume (with $\hbar = 1$) that

$$[\Phi_a(P,Q), \Phi_b(P,Q)] = ic_{ab}{}^c\,\Phi_c(P,Q)\ , \tag{5}$$
$$[\Phi_a(P,Q), \mathcal{H}(P,Q)] = ih_a{}^b\,\Phi_b(P,Q)\ , \tag{6}$$

where Φ_a and $\mathcal{H}$ denote self-adjoint constraint and Hamiltonian operators, respectively. Following Dirac [1], we adopt the quantization prescription given by

$$i\dot{W}(P,Q) = [W(P,Q), \mathcal{H}(P,Q)] \tag{7}$$

where W denotes a general function of the kinematical operators $\{Q^j\}$ and $\{P_j\}$ which are taken as a self-adjoint, irreducible representation of the commutation rules $[Q^j, P_k] = i\delta^j_k \mathbb{1}$, with all other commutators vanishing. The

equations of motion hold for all time t, say $0 < t < T$. On the other hand, the conditions

$$\Phi_a(P, Q)|\psi\rangle_{phys} = 0 \tag{8}$$

to select the physical Hilbert space are imposed only at time $t = 0$ as the analog of the initial value equation; the quantum equations of motion ensure that the constraint conditions are fulfilled for all time.

The procedure of Dirac has potential difficulties if zero lies in the continuous spectrum of the constraint operators for in that case there are no normalizable solutions of the constraint condition. We face the same problem, of course, and our resolution is discussed below.

1.4 Reduction first: Standard path integral quantization

Faddeev [2] has given a path integral formulation in the case of closed first-class constraint systems as follows. The formal path integral

$$\begin{aligned}\int \exp\{i\textstyle\int_0^T [p_j\dot{q}^j - H(p,q) - \lambda^a\phi_a(p,q)]\,dt\}\,\mathcal{D}p\,\mathcal{D}q\,\mathcal{D}\lambda \\ = \int \exp\{i\textstyle\int_0^T [p_j\dot{q}^j - H(p,q)]\,dt\}\,\delta\{\phi(p,q)\}\,\mathcal{D}p\,\mathcal{D}q\end{aligned} \tag{9}$$

may well encounter divergences in the remaining integrals. Therefore, subsidiary conditions in the form $\chi^a(p,q) = 0$, $1 \leq a \leq A$, are imposed picking out (ideally) one gauge equivalent point per gauge orbit, and in addition a factor in the form of the Faddeev-Popov determinant is introduced to formally preserve canonical covariance. The result is the path integral

$$\int \exp\{i\textstyle\int_0^T [p_j\dot{q}^j - H(p,q)]\,dt\}\,\delta\{\chi(p,q)\}\,\det(\{\chi^a,\phi_b\})\delta\{\phi(p,q)\}\,\mathcal{D}p\,\mathcal{D}q\,. \tag{10}$$

This result may also be expressed as

$$\int \exp\{i\textstyle\int_0^T [p^*_j\dot{q}^{*j} - H^*(p^*,q^*)]\,dt\}\,\mathcal{D}p^*\,\mathcal{D}q^*\;, \tag{11}$$

namely, as a path integral over a reduced phase space in which the δ-functionals have been used to eliminate $2A$ integration variables.

The final expression generally involves an integral over a non-Euclidean phase space for which the conventional definition of the path integral is typically ill defined. Thus this widely used prescription is not without its difficulties.

1.5 Quantization first $\not\equiv$ reduction first

The two schemes illustrated in the preceding sections are different in principle. In the initial case, one quantizes *first* and reduces *second;* in the latter case, one reduces *first* and quantizes *second.* For certain systems the results of these different procedures are the same, but that is not universally the case, as we now proceed to illustrate.

Let us consider the example of a single degree of freedom specified by the classical action

$$I = \int [p\dot{q} - \lambda(p^2 + q^4 - E)]\, dt \; . \tag{12}$$

Observe that the classical Hamiltonian vanishes and there is a single constraint. The question we pose is: For what values of E, $E > 0$, is the quantum theory nontrivial?

On the one hand, according to the procedure of Dirac, the physical Hilbert space is either empty or one-dimensional, spanned by the nonvanishing eigenvector $|\psi_n\rangle$ that satisfies

$$(P^2 + Q^4)|\psi_n\rangle = E_n\, |\psi_n\rangle \; , \tag{13}$$

for E_n one of the purely discrete eigenvalues for the "Hamiltonian" $P^2 + Q^4$.

On the other hand, the procedure of Faddeev leads initially to

$$\int e^{i\int p\, dq}\, \delta\{p^2 + q^4 - E\}\, \mathcal{D}p\, \mathcal{D}q \; . \tag{14}$$

Next, we fix a gauge, e.g., $p = 0$, in which case the reduced phase space propagator is given by

$$\begin{aligned} \int e^{i\int p\, dq}\, \delta\{p^2 + q^4 - E\}\, \Pi(4q^3)\, \delta\{p\}\, \mathcal{D}p\, \mathcal{D}q \\ = 0 \; , \end{aligned} \tag{15}$$

which vanishes due to cancellation between the term with $q > 0$ and the term with $q < 0$. Note that the symbol Π denotes a formal multiplication over all time points. An alternative evaluation may be given if we allow only the term with $q > 0$, which is achieved by instead using

$$\begin{aligned}
&\int e^{i\int p\,dq}\,\delta\{p^2+q^4-E\}\,\delta\{p\}\,\mathcal{D}p\,\mathcal{D}q^4 \\
&\quad = \int \delta\{q^4-E\}\,\mathcal{D}q^4 \\
&\quad = 1\,. \qquad (16)
\end{aligned}$$

Either of these choices imposes *no* restriction on E whatsoever. Ignoring the nonphysical nature of the variables involved, one might possibly impose the condition

$$\oint p\,dq = 2\pi n\,, \qquad (17)$$

leading to a Bohr-Sommerfeld spectrum, which for this problem is incorrect. (The reader is encouraged to examine alternative choices of gauge.)

Remark: It is instructive in this example to note that the Faddeev-Popov determinant $\Delta = \Pi(4q^3)$ and the reduced phase space is the single point $(p, q) = (0, E^{1/4})$. The point $(p, q) = (0, -E^{1/4})$ corresponds to a Gribov copy.

Clearly, in this case, reduction before quantization has led to the wrong result. Some workers may assert that such errors are merely "order $\hbar$ corrections". Although true, this argument cannot be used to defend the general procedure since the role of a quantization procedure, after all, should be to determine the *correct* spectrum for a specific problem, not a spectrum that is potentially incorrect even in its *leading order*. Examples of other work which arrive at the same conclusion are given in [11].

1.6 Outline of the remaining sections

In the following section, Sec. 2, we present an overview of the *projection operator approach* to constrained system quantization with an emphasis on coherent-state representations. Section 3 deals with coherent-state path integrals without gauge fixing for closed first-class constrained systems. Extensions to general constraints such as open first-class or second-class systems

are the subject of Sec. 4. Section 5 is devoted to selected examples of first-class systems, while Sec. 6 concentrates on two rather special applications. Finally, in Sec. 7 we comment on some other applications of the projection operator approach that have not been discussed in this paper.

2 OVERVIEW OF THE PROJECTION OPERATOR APPROACH TO CONSTRAINED SYSTEM QUANTIZATION

2.1 Coherent states

Canonical quantization is consistent only for Cartesian phase space coordinates [7], and we assume that our original and unconstrained set of classical dynamical variables fulfill that condition. Then, for each classical coordinate q^j and momentum p_j, $1 \leq j \leq J$, we may introduce associated self-adjoint canonical operators Q^j and P_j, acting in a separable Hilbert space $\mathfrak{H}$, and which satisfy, in units where $\hbar = 1$, the canonical commutation relations $[Q^j, P_k] = i\delta^j_k\,1\!\!1$, with all other commutation relations vanishing. With the fiducial vector $|0\rangle \in \mathfrak{H}$ a suitable normalized state—typically the ground state of a (unit-frequency) harmonic oscillator (but not always!)—we introduce the *canonical coherent states* (see, e.g., [12, 13])

$$|p,q\rangle \equiv e^{-iq^jP_j}\,e^{ip_jQ^j}\,|0\rangle\ , \tag{18}$$

for all $(p,q) \in \mathrm{I\!R}^{2J}$, where $p = \{p_j\}$ and $q = \{q^j\}$. These states admit a resolution of unity in the form [14]

$$1\!\!1 = \int |p,q\rangle\langle p,q|\,d\mu(p,q)\ , \qquad d\mu(p,q) \equiv d^Jp\,d^Jq/(2\pi)^J\ , \tag{19}$$

integrated over $\mathrm{I\!R}^{2J}$.

The unit operator resides in the Hilbert space $\mathfrak{H}$ of the unconstrained system. We may conveniently represent this Hilbert space as follows. We first introduce the *reproducing kernel* $\langle p'',q''|p',q'\rangle$ as the overlap matrix element between any two coherent states. This expression is a bounded, continuous function that characterizes a (reproducing kernel Hilbert space) representation of $\mathfrak{H}$ appropriate to the unconstrained system as follows. A dense set of

vectors in the associated functional Hilbert space is given by vectors of the form

$$\psi(p,q) \equiv \langle p,q|\psi\rangle = \sum_{l=1}^{L} \alpha_l \, \langle p,q|p_{(l)},q_{(l)}\rangle \ , \tag{20}$$

for arbitrary sets $\{\alpha_l\}$ and $\{p_{(l)},q_{(l)}\}$ with $L<\infty$. The inner product of two such vectors is given by

$$(\psi,\xi) \equiv \sum_{l,m=1}^{L,M} \alpha_l^* \, \beta_m \, \langle p_{(l)},q_{(l)}|\overline{p}_{(m)},\overline{q}_{(m)}\rangle \tag{21}$$

$$= \int \psi(p,q)^* \xi(p,q) \, d\mu(p,q) \ , \tag{22}$$

where ξ is a second function defined in a manner analogous to ψ. A general vector in the functional Hilbert space is defined by a Cauchy sequence of such vectors, and all such vectors are given by bounded, continuous functions. The first form of the inner product applies in general only to vectors in the dense set, while the second form of the inner product holds for arbitrary vectors in the Hilbert space. We shall have more to say below regarding reproducing kernels and reproducing kernel Hilbert spaces.

2.2 Constraints

Now suppose we introduce constraints into the quantum theory [8]. In particular, we assume that $\mathbb{E}$ denotes a *projection operator onto the constraint subspace*, i.e., the subspace on which the quantum constraints are satisfied (in a sense to be defined below), and which is called the physical Hilbert space $\mathfrak{H}_{phys} \equiv \mathbb{E}\,\mathfrak{H}$. Later we shall discuss examples of $\mathbb{E}$. Hence, if $|\psi\rangle \in \mathfrak{H}$ denotes a general vector in the original (unconstrained) Hilbert space, the vector $\mathbb{E}|\psi\rangle \in \mathfrak{H}_{phys}$ represents its component within the physical subspace. As a Hilbert space, the physical subspace also admits a functional representation by means of a reproducing kernel which may be taken as $\langle p'',q''|\mathbb{E}|p',q'\rangle$. In the same manner as before, it follows that a dense set of vectors in $\mathfrak{H}_{phys}$ is given by functions of the form

$$\psi(p,q) \equiv \langle p,q|\mathbb{E}|\psi\rangle = \sum_{l=1}^{L} \alpha_l \, \langle p,q|\mathbb{E}|p_{(l)},q_{(l)}\rangle \ , \tag{23}$$

for arbitrary sets $\{\alpha_l\}$ and $\{p_{(l)}, q_{(l)}\}$ with $L < \infty$. The inner product of two such vectors is given by

$$(\psi, \xi) \equiv \sum_{l,m=1}^{L,M} \alpha_l^* \, \beta_m \, \langle p_{(l)}, q_{(l)} | \mathbb{E} | \overline{p}_{(m)}, \overline{q}_{(m)} \rangle$$
$$= \int \psi(p,q)^* \xi(p,q) \, d\mu(p,q) \; . \tag{24}$$

Again, a general vector in the functional Hilbert space is defined by means of a Cauchy sequence, and all such vectors are given by bounded, continuous functions. Note well, in the case illustrated, that even though $\mathbb{E}\,\mathfrak{H} \subset \mathfrak{H}$, the functional representation of the unconstrained and the constrained Hilbert spaces are *identical*, namely by functions of $(p,q) \in \mathbb{R}^{2J}$, and the form of the inner product is *identical* in the two cases. This situation holds even if $\mathfrak{H}_{phys}$ is one dimensional!

The relation between the self-adjoint constraint operators Φ_a, $1 \leq a \leq A$, $A < \infty$, and the projection operator $\mathbb{E}$ may take several different forms. Unless otherwise specified, we shall assume that $\Sigma_a \Phi_a^2$ is self adjoint and that

$$\mathbb{E} = \mathbb{E}(\Sigma_a \Phi_a^2 \leq \delta(\hbar)^2) \; , \tag{25}$$

where $\delta = \delta(\hbar)$ (*not* a Dirac δ-function!) is a regularization parameter which is chosen in accord with rules to be discussed below.

2.3 Dynamics for first-class systems

Suppose further that the Hamiltonian $\mathcal{H}$ respects the first-class character of the constraints. It follows in this case that $[\mathbb{E}, \mathcal{H}] = 0$ or stated otherwise that

$$e^{-i\mathcal{H}t}\mathbb{E} \equiv \mathbb{E}\, e^{-i\mathcal{H}t}\mathbb{E} \equiv \mathbb{E}\, e^{-i(\mathbb{E}\mathcal{H}\mathbb{E})\,t}\mathbb{E} \; . \tag{26}$$

Dynamics in the physical subspace is then fully determined by the propagator on $\mathfrak{H}_{phys}$, which is given in the relevant functional representation by

$$\langle p'', q'' | e^{-i\mathcal{H}t}\mathbb{E} | p', q' \rangle \; . \tag{27}$$

In (27) we have achieved a fully gauge invariant propagator without having to reduce the range or even the number of the original classical variables nor change the original form of the inner product on the functional Hilbert space representation. Any observable $\mathcal{O}$—$\mathcal{H}$ included—satifies $[\mathbb{E}, \mathcal{O}] = 0$, and relations similar to (26) follow with $\mathcal{H}$ replaced by $\mathcal{O}$.

2.4 Zero in the continuous spectrum

The foregoing scenario has assumed that the appropriate $\mathfrak{H}_{phys}$ is given by means of a projection operator $\mathbb{E}$ acting on the original Hilbert space. This situation holds true whenever the set of quantum constraints admits zero as a common point in their discrete spectrum; in that case $\mathbb{E}$ defines the subspace where the constraints all vanish. That situation may not always hold true, but even in case zero lies in the continuous spectrum for some or all of the constraints, a suitable result may generally be given by matrix elements of a sequence of rescaled projection operators, say $c_\delta\,\mathbb{E}$, $c_\delta > 0$, as $\delta \to 0$. Specifically, we consider the limit of a sequence of reproducing kernels $c_\delta\,\langle p'', q''|\mathbb{E}|p', q'\rangle$, which—if the limit is a nonvanishing continuous function—defines a new reproducing kernel, and thereby a new reproducing kernel Hilbert space, within which the appropriate constraints are fulfilled. In such a limit certain variables may cease to be relevant and as a consequence the local integral representation of the inner product, if any, may require modification. On the other hand, the definition of the inner product by sums involving the reproducing kernel will always hold. We refer to the result of such a limiting operation as a *reduction of the reproducing kernel.* A simple example should help clarify what we mean by a reduction of the reproducing kernel.

Consider the example

$$\langle p'', q''|\mathbb{E}|p', q'\rangle$$
$$= \pi^{-1/2} \int_{-\delta}^{\delta} \exp[-\tfrac{1}{2}(k - p'')^2 + ik(q'' - q') - -\tfrac{1}{2}(k - p')^2]\, dk\,, \quad (28)$$

where $\mathbb{E} = \mathbb{E}(P^2 \leq \delta^2)$, which defines a reproducing kernel for any $\delta > 0$ that corresponds to an infinite dimensional Hilbert space. (If $\delta = \infty$ the result is the usual canonical coherent state overlap and characterizes the unconstrained Hilbert space.) If we take the limit of the expression as it

stands as $\delta \to 0$, the result will vanish. What we need to do is extract the *germ* of the projection operator as we let δ go to zero. Therefore, let us first multiply this expression by $\pi^{1/2}/(2\delta)$ [c_δ in this case] and take the limit $\delta \to 0$. The result is the expression

$$\mathcal{K}(p''; p') = e^{-\frac{1}{2}(p''^2 + p'^2)} \,, \tag{29}$$

which has become a reproducing kernel that characterizes a *one*-dimensional Hilbert space with every functional representative proportional to $\chi_o(p) \equiv \exp(-p^2/2)$. This one-dimensional Hilbert space representation also admits a local integral representation for the inner product given by

$$(\chi, \chi) = \textstyle\int |\chi(p)|^2 \, dp/\sqrt{\pi} \,. \tag{30}$$

In the present case, it is clear that one may reduce the reproducing kernel even further by choosing $p = c$, an arbitrary but fixed constant. This kind of reduction—in which the latter reproducing kernel Hilbert space is equivalent to the former reproducing kernel Hilbert space—is analogous to choosing a gauge in the classical theory. We shall see another example of this latter kind of reduction later.

The example presently under discussion is also an important one inasmuch as it illustrates how a constraint operator with its zero lying in the continuous spectrum is dealt with in the coherent-state, projection-operator approach. Some other approaches to deal with the problem of zero in the continuous spectrum may be traced from [15].

2.5 Alternative view of continuous zeros

If $\delta \ll 1$ in (28), then it may be approximately evaluated as

$$\begin{aligned} &\langle p'', q'' | \mathbb{E} | p', q' \rangle \\ &\qquad = \pi^{-1/2} \delta \, e^{-\frac{1}{2}(p''^2 + p'^2)} \, \frac{\sin[\delta(q'' - q')]}{\delta(q'' - q')} + O(\delta^2) \,. \end{aligned} \tag{31}$$

When $\delta = 10^{-1000}$, or some other extremely tiny factor, it is clear that for all practical purposes it is sufficient to accept just the first term in (31), ignoring the term $O(\delta^2)$, as the "reduced" reproducing kernel. The resultant expression is indeed a proper reproducing kernel for which inner products

are given with the full set of integration variables and the normal integration range. So long as q values are "normal sized", e.g., $|q| < 10^{500}$ in the present case, there is no practical distinction between the space of functions generated by (29) and that generated by (31). In other words, if δ is chosen extremely close to zero, but still positive, it is not actually necessary to take the limit $\delta \rightarrow 0$ in order to do practical calculations. Even though this is the case, we shall for the most part in the examples we study take a full reduction by first rescaling the reproducing kernel (by an appropriate factor c_δ) and then taking the limit $\delta \rightarrow 0$.

3 COHERENT STATE PATH INTEGRALS WITHOUT GAUGE FIXING

As introduced above, canonical coherent states may be defined by the relation

$$|p, q\rangle \equiv e^{-iq^j P_j}\, e^{ip_j Q^j}\, |0\rangle \;, \tag{32}$$

for all (p, q), where the fiducial vector $|0\rangle$ traditionally denotes a normalized, unit frequency, harmonic oscillator ground state, and the coherent states admit a resolution of unity in the form

$$\mathbb{1} = \int |p, q\rangle\langle p, q|\, d\mu(p, q) \;, \qquad d\mu(p, q) \equiv d^J\!p\, d^J\!q/(2\pi)^J \;, \tag{33}$$

where the integration is over $\mathbb{R}^{2J}$. Note that the integration domain and the form of the measure are unique.

Based on such coherent states, we introduce the upper symbol for a general operator $\mathcal{H}(P, Q)$,

$$H(p, q) \equiv \langle p, q|\mathcal{H}(P, Q)|p, q\rangle = \langle p, q| : H(P, Q) : |p, q\rangle \tag{34}$$

which is related to the normal-ordered form as shown. (N.B. Some workers would call $H(p, q)$ the lower symbol.) If $\mathcal{H}(P, Q)$ denotes the quantum Hamiltonian, then we shall adopt $H(p, q)$ as the classical Hamiltonian. We also note that an important one-form generated by the coherent states is given by $i\langle p, q|d|p, q\rangle = p_j\, dq^j$.

Using these quantities, and the time ordering operator T, the coherent state path integral for the propagator generated by the time-dependent

Hamiltonian $\mathcal{H}(P,Q)+\lambda^a(t)\Phi_a(P,Q)$ is readily given by

$$
\begin{aligned}
&\langle p'',q''|\mathsf{T}e^{-i\int_0^T[\mathcal{H}(P,Q)+\lambda^a(t)\Phi_a(P,Q)]\,dt}|p',q'\rangle \\
&\quad=\lim_{\epsilon\to 0}\int\prod_{l=0}^{N}\langle p_{l+1},q_{l+1}|e^{-i\epsilon(\mathcal{H}+\lambda^a_l\Phi_a)}\,|p_l,q_l\rangle\prod_{l=1}^{N}d\mu(p_l,q_l) \\
&\quad=\int\exp\{i{\textstyle\int}[i\langle p,q|(d/dt)|p,q\rangle-\langle p,q|\mathcal{H}+\lambda^a(t)\Phi_a|p,q\rangle]\,dt\}\,\mathcal{D}\mu(p,q) \\
&\quad=\mathcal{M}\int\exp\{i{\textstyle\int}[p_j\dot q^j-H(p,q)-\lambda^a(t)\phi_a(p,q)]\,dt\}\,\mathcal{D}p\,\mathcal{D}q\;. \qquad (35)
\end{aligned}
$$

Here, in the second line, we have set $\epsilon\equiv T/(N+1)$, made a Trotter-product like approximation to the evolution operator, repeatedly inserted the resolution of unity, and set $p_{N+1},q_{N+1}=p'',q''$ and $p_0,q_0=p',q'$. In the third and fourth lines we have formally interchanged the continuum limit and the integrations, and written for the integrand the form it would assume for continuous and differentiable paths ($\mathcal{M}$ denotes a formal normalization constant). The result evidently depends on the chosen form of the functions $\{\lambda^a(t)\}$.

3.1 Enforcing the quantum constraints

Let us next introduce the quantum analog of the initial value equation. For simplicity we assume that the constraint operators form a compact group; more general situations are dealt with below. In that case

$$
\mathbb{E}\equiv{\textstyle\int}e^{-i\xi^a\Phi_a(P,Q)}\,\delta\xi=\mathbb{E}(\Phi_a=0,\;\;1\le a\le A)=\mathbb{E}(\Sigma_a\Phi_a^2=0) \qquad (36)
$$

defines a *projection operator* onto the subspace for which $\Phi_a=0$ provided that $\delta\xi$ denotes the normalized, $\int\delta\xi=1$, group invariant measure. It follows from (36) that

$$
e^{-i\tau^a\Phi_a}\mathbb{E}=\mathbb{E}\;. \qquad (37)
$$

We now project the propagator (35) onto the quantum constraint subspace which leads to the following set of relations

$$
\int\langle p'',q''|\mathsf{T}e^{-i\int[\mathcal{H}+\lambda^a(t)\Phi_a]\,dt}\,|\overline{p}',\overline{q}'\rangle\langle\overline{p}',\overline{q}'|\mathbb{E}\,|p',q'\rangle\,d\mu(\overline{p}',\overline{q}')
$$

$$
\begin{aligned}
&= \langle p'', q''|\mathsf{T}e^{-i\int[\mathcal{H}+\lambda^a(t)\Phi_a]\,dt}\,\mathbb{E}\,|p', q'\rangle \\
&= \lim\,\langle p'', q''|[\overleftarrow{\prod_l}(e^{-i\epsilon\mathcal{H}}e^{-i\epsilon\lambda^a_l\Phi_a})]\,\mathbb{E}\,|p', q'\rangle \\
&= \langle p'', q''|e^{-iT\mathcal{H}}e^{-i\tau^a\Phi_a}\,\mathbb{E}\,|p', q'\rangle \\
&= \langle p'', q''|e^{-iT\mathcal{H}}\,\mathbb{E}\,|p', q'\rangle\ ,
\end{aligned}
\tag{38}
$$

where τ^a incorporates the functions λ^a as well as the structure parameters ${c_{ab}}^c$ and ${h_a}^b$. Alternatively, this expression has the formal path integral representation

$$
\int \exp\{i{\textstyle\int}[p_j\dot{q}^j - H(p,q) - \lambda^a(t)\phi_a(p,q)]\,dt - i\xi^a\phi_a(p', q')\}\,\mathcal{D}\mu(p,q)\,\delta\xi\ . \tag{39}
$$

On comparing (35) and (39), we observe that *after projection onto the quantum constraint subspace the propagator is entirely independent of the choice of the Lagrange multiplier functions. In other words, the projected propagator is gauge invariant.*

We may also express the physical (projected) propagator in a more general form, namely,

$$
\begin{aligned}
&\int \exp\{i{\textstyle\int}[p_j\dot{q}^j - H(p,q) - \lambda^a(t)\phi_a(p,q)]\,dt\}\,\mathcal{D}\mu(p,q)\,\mathcal{D}C(\lambda) \\
&\qquad = \langle p'', q''|e^{-iT\mathcal{H}}\,\mathbb{E}\,|p', q'\rangle
\end{aligned}
\tag{40}
$$

provided that $\int\mathcal{D}C(\lambda) = 1$ and that such an average over the functions $\{\lambda^a(t)\}$ introduces (at least) one factor $\mathbb{E}$.

3.2 Reproducing kernel Hilbert spaces

The coherent-state matrix elements of $\mathbb{E}$ define a fundamental kernel

$$
\mathcal{K}(p'', q''; p', q') \equiv \langle p'', q''|\mathbb{E}|p', q'\rangle\ , \tag{41}
$$

which is a bounded, continuous function for any projection operator $\mathbb{E}$, especially including the unit operator. It follows that $\mathcal{K}(p'', q''; p', q')^* = \mathcal{K}(p', q'; p'', q'')$ as well as

$$
\sum_{k,l=1}^{K} \alpha_k^*\alpha_l\mathcal{K}(p_k, q_k; p_l, q_l) \geq 0 \tag{42}
$$

for all sets $\{\alpha_k\}$, $\{(p_k, q_k)\}$, and all $K < \infty$. The last relation is an automatic consequence of the complex conjugate property and the fact that

$$\mathcal{K}(p'', q''; p', q') = \int \mathcal{K}(p'', q''; p, q)\, \mathcal{K}(p, q; p', q')\, d\mu(p, q) \tag{43}$$

holds in virtue of the coherent state resolution of unity and the properties of $\mathbb{E}$. As noted earlier, the function $\mathcal{K}$ is called the *reproducing kernel* and the Hilbert space it engenders is termed a *reproducing kernel Hilbert space* [16]. A dense set of elements in the Hilbert space is given by functions of the form

$$\psi(p, q) = \sum_{k=1}^{K} \alpha_k \mathcal{K}(p, q; p_k, q_k)\ , \tag{44}$$

and the inner product of this function has two equivalent forms given by

$$\begin{aligned} (\psi, \psi) &= \sum_{k,l=1}^{K} \alpha_k^* \alpha_l \mathcal{K}(p_k, q_k; p_l, q_l) & (45)\\ &= \textstyle\int \psi(p, q)^* \psi(p, q)\, d\mu(p, q)\ . & (46) \end{aligned}$$

The inner product of two distinct functions may be determined by polarization of the norm squared [17]. Clearly, the entire Hilbert space is characterized by the reproducing kernel $\mathcal{K}$. Change the kernel $\mathcal{K}$ and one changes the representation of the Hilbert space. Following a suitable limit of the kernel $\mathcal{K}$, it is even possible to change the *dimension* of the Hilbert space, as already illustrated earlier.

3.3 Reduction of the reproducing kernel

Suppose the reproducing kernel depends on a number of variables and additional parameters. We can generate new reproducing kernels from a given kernel by a variety of means. For example, the expressions

$$\begin{aligned} \mathcal{K}_1(p''; p') &= \mathcal{K}(p'', c; p', c)\ , & (47)\\ \mathcal{K}_2(p''; p') &= \textstyle\int f(q'')^* f(q') \mathcal{K}(p'', q''; p', q')\, dq''\, dq'\ , & (48)\\ \mathcal{K}_3(p'', q''; p', q') &= \lim \mathcal{K}(p'', q''; p', q') & (49) \end{aligned}$$

each generate a new reproducing kernel provided the resultant function remains continuous. In general, however, the inner product in the Hilbert space generated by the new reproducing kernel is only given by an analog of (21) and not by (22), although frequently some sort of local integral representation for the inner product may exist.

Let us offer an example of the reduction of a reproducing kernel that is a slight generalization of the earlier example. Let the expression

$$\langle p'', q''|\mathbb{E}|p', q'\rangle \equiv \\ \pi^{-J/2}\int_{-\delta}^{\delta}\cdots\int_{-\delta}^{\delta}\exp[-\tfrac{1}{2}(k-p'')^2+ik\cdot(q''-q')-\tfrac{1}{2}(k-p')^2]\,d^Jk \quad (50)$$

denote a reproducing kernel for any $\delta > 0$. In the present case it follows that $\mathbb{E} \equiv \Pi_{j=1}^{J}\mathbb{E}(-\delta \leq P_j \leq \delta)$. When $\delta \to 0$, then (50) vanishes. However, if we first multiply by δ^{-J}—or more conveniently by $\pi^{J/2}(2\delta)^{-J}$—before taking the limit, the result becomes

$$\lim_{\delta\to 0}\pi^{J/2}(2\delta)^{-J}\langle p'', q''|\mathbb{E}|p', q'\rangle = \exp(-\tfrac{1}{2}\,p''^2)\exp(-\tfrac{1}{2}\,p'^2)\ , \quad (51)$$

which is continuous and therefore denotes the reproducing kernel for some Hilbert space. Note that the classical variables q'' and q' have disappeared, which on reference to (32) implies that all "$P_j = 0$". In the present example, the resultant Hilbert space is one dimensional, and the inner product may be given either by a sum as in (21) involving the p variables alone or by a local integral representation now using the measure $\pi^{-J/2}\,d^Jp$, namely,

$$(\chi, \chi) = \textstyle\int|\chi(p)|^2\pi^{-J/2}\,d^Jp\ . \quad (52)$$

This example illustrates the case where the constraints are "$P_j = 0$", for all j, a situation where zero lies in the continuous spectrum.

We may also use this example to illustrate how *several* constraints may be replaced by a *single* constraint. The several constraints "$P_j = 0$", for all j, were first approximated by the regularized constraints $P_j^2 \leq \delta^2$, $\delta > 0$, for all j. Alternatively, we may also regularize the constraints in the form $\Sigma_j P_j^2 \leq \delta^2$. Furthermore, if we use $\mathbb{E} = \mathbb{E}(\Sigma_j P_j^2 \leq \delta^2)$, then it is clear that a new prefactor, also proportional to δ^{-J}, can be chosen so that (51) again emerges as $\delta \to 0$.

3.4 Single regularized constraints

Clearly, the *set* of real classical constraints $\phi_a = 0$, $1 \le a \le A$, is equivalent to the *single* classical constraint $\Sigma_a \phi_a^2 = 0$. Likewise, the set of (idealized) quantum constraints "$\Phi_a |\psi\rangle_{phys} = 0$", $1 \le a \le A$, where each Φ_a is self adjoint, is equivalent to the single (idealized) quantum constraint "$\Sigma_a \Phi_a^2 |\psi\rangle_{phys} = 0$", where we further assume that $\Sigma_a \Phi_a^2$ is a self-adjoint operator. In general, however, the only solution of the idealized quantum constraint is the zero vector, $|\psi\rangle_{phys} = 0$.

To overcome this difficulty, we relax the idealized quantum constraint and instead generally adopt the regularized form of the constraint given by $|\psi\rangle_{phys} \in \mathcal{H}_{phys} \equiv \mathbb{E}\,\mathcal{H}$, where

$$\mathbb{E} = \mathbb{E}(\Sigma_a \Phi_a^2 \le \delta(\hbar)^2) \ . \tag{53}$$

Here $\delta(\hbar)$ is a regularization parameter and the inequality means that in a spectral resolution of $\Sigma_a \Phi_a^2 \equiv \int_0^\infty \lambda \, dE(\lambda)$ that

$$\mathbb{E} \equiv \int_0^{\delta(\hbar)^2} dE(\lambda) = E(\delta(\hbar)^2) \ . \tag{54}$$

Let us examine three basic examples.

First, let zero be in the discrete spectrum of $\Sigma_a \Phi_a^2$. Then, it follows that there exists a $\delta_1(\hbar)^2$ such that for all $\delta(\hbar)^2$, $0 < \delta(\hbar)^2 < \delta_1(\hbar)^2$, then $\mathbb{E}(\Sigma_a \Phi_a^2 \le \delta(\hbar)^2) = \mathbb{E}(\Sigma_a \Phi_a^2 = 0)$.

Second, if $\Sigma_a \Phi_a^2$ has its zero in the continuum, then $\mathbb{E}(\Sigma_a \Phi_a^2 \le \delta^2)$ is infinite dimensional for all $\delta > 0$, but $\mathbb{E}$ vanishes weakly as $\delta \to 0$. For such cases we consider $c_\delta \mathbb{E}$ and choose the sequence c_δ to weakly extract the *germ* of $\mathbb{E}$ as $\delta \to 0$, just as in the examples illustrated above.

Third, in a case to be studied later, suppose that zero is *not* in the spectrum of the operator $\Sigma_a \Phi_a^2$. Since $\Sigma_a \phi_a^2 = 0$ classically, it follows that spectral values of $\Sigma_a \Phi_a^2$ are $o(\hbar^0)$ close to zero. A relevant example discussed later is where $\Phi_1 = P$ and $\Phi_2 = Q$. Then $\mathbb{E}(P^2 + Q^2 \le \hbar) = |0\rangle\langle 0|$ is a one-dimensional projection operator onto the harmonic oscillator ground state $|0\rangle$. Observe in this case that $\delta(\hbar)^2 = \hbar$, which vanishes when $\hbar \to 0$; note also that we cannot reduce this parameter further since $\mathbb{E}(P^2 + Q^2 < \hbar) \equiv 0$. Thus, in some cases, whether we use "$\le$" or "$<$" in the inequality defining the projection operator can make a real difference.

The three types of examples discussed above illustrate three qualitatively different behaviors possible for the projection operator $\mathbb{E}$. As we proceed, we shall find the use of a single regularized constraint will be an important unifying principle in treating the most general multiple constraint situation imaginable.

3.5 Basic first-class constraint example

Consider the system with two degrees of freedom, a vanishing Hamiltonian, and a single constraint, characterized by the action

$$I = \textstyle\int [\frac{1}{2}(p_1\dot{q}_1 - q_1\dot{p}_1 + p_2\dot{q}_2 - q_2\dot{p}_2) - -\lambda(q_2p_1 - p_2q_1)]\, dt \;, \tag{55}$$

where for notational convenience we have lowered the index on the q variables. Note that we have chosen a different form for the kinematic part of the action which amounts to a change of phase for the coherent states, and in particular a factor of $e^{ipq/2}$ has been introduced on the right side of (18), or, equivalently, both generators appear in the same exponent. It follows that

$$\begin{aligned} \mathcal{M}\int \exp\{i\textstyle\int [\frac{1}{2}(p_1\dot{q}_1 - q_1\dot{p}_1 + p_2\dot{q}_2 - q_2\dot{p}_2) - \lambda(q_2p_1 - p_2q_1)]\, dt\} \\ \times \mathcal{D}p\,\mathcal{D}q\,\mathcal{D}C(\lambda) \\ = \langle p'', q''|\mathbb{E}\,|p', q'\rangle \;, \end{aligned} \tag{56}$$

where we choose

$$\mathbb{E} = (2\pi)^{-1}\int_0^{2\pi} e^{-i\xi(Q_2P_1 - P_2Q_1)}\, d\xi = \mathbb{E}\,(L_3 = 0) \;. \tag{57}$$

Based on the fact [12] that

$$\langle p'', q''|p', q'\rangle = \exp(-\tfrac{1}{2}|z_1''|^2 - \tfrac{1}{2}|z_2''|^2 + z_1''^{*}z_1' + z_2''^{*}z_2' - \tfrac{1}{2}|z_1'|^2 - \tfrac{1}{2}|z_2'|^2) \;, \tag{58}$$

where $z_1' \equiv (q_1' + ip_1')/\sqrt{2}$, etc., it is straightforward to show that

$$\begin{aligned} \langle p'', q''|\mathbb{E}\,|p', q'\rangle = \exp(-\tfrac{1}{2}|z_1''|^2 - \tfrac{1}{2}|z_2''|^2 - \tfrac{1}{2}|z_1'|^2 - \tfrac{1}{2}|z_2'|^2) \\ \times I_0(\,(z_1''^{*2} + z_2''^{*2})^{1/2}(z_1'^2 + z_2'^2)^{1/2}\,) \;, \end{aligned} \tag{59}$$

with I_0 a standard Bessel function. We emphasize again that although the Hilbert space has been strictly reduced by the introduction of $\mathbb{E}$, the reproducing kernel (59) leads to a reproducing kernel Hilbert space with an inner product having the same number of integration variables and domain of integration as in the unconstrained case.

4 APPLICATION TO GENERAL CONSTRAINTS

4.1 Classical considerations

When dealing with a general constraint situation it will typically happen that the self-consistency of the equations of motion may determine some or all of the Lagrange multipliers in order for the system to remain on the classical constraint hypersurface. For example, if the Hamiltonian attempts to force points initially lying on the constraint hypersurface to leave that hypersurface, then the Lagrange multipliers must supply the necessary forces for the system to remain on the constraint hypersurface.

We may elaborate on this situation as follows. Since $\phi_a(p,q) = 0$ for all a defines the constraint hypersurface, it is also necessary, for all a, that

$$\dot{\phi}_a(p,q) \equiv \{\phi_a(p,q), H(p,q)\} + \lambda^b(t)\{\phi_a(p,q), \phi_b(p,q)\} \equiv 0 \tag{60}$$

also holds on the constraint hypersurface. If the Poisson brackets fulfill the conditions given in (3) and (4), then it follows that $\dot{\phi}_a(p,q) \equiv 0$ on the constraint hypersurface for any choice of the Lagrange multipliers $\{\lambda^a(t)\}$. This is the case for first-class constraints, and to obtain specific solutions to the dynamical equations it is necessary to specify some choice of the Lagrange multipliers, i.e., to select a gauge. However, if (3), or (3) and (4) do *not* hold on the constraint hypersurface, the situation changes. For example, let us first assume that (4) holds but that

$$\Delta_{ab}(p,q) \equiv \{\phi_a(p,q), \phi_b(p,q)\} \tag{61}$$

is a *non*singular matrix on the constraint hypersurface. In this case it follows that we must choose $\lambda^a(t) \equiv 0$ for all a to satisfy (60). More generally, we must choose

$$\lambda^a(t) \equiv -(\Delta^{-1}(p,q))^{ab}\,\{\phi_b(p,q),\, H(p,q)\} \tag{62}$$

in order that (60) will be satisfied. When the Lagrange multipliers are not arbitrary but rather must be specifically chosen in order to keep the system on the constraint hypersurface, then we say that we deal with second-class constraints. Of course, there are also intermediate situations where part of

the constraints are first class while some are second class; in this case the matrix $\Delta_{ab}(p,q)$ would be singular but would have a nonzero rank on the constraint hypersurface.

Remark: It is useful to also imagine solving the differential equation (60) as a computer might do it, namely, by an iteration procedure. In particular, we could imagine evolving by a small time step ϵ by the first (Hamiltonian) term, then using the second (constraint) term to choose λ^a at that moment to force the system back onto the constraint hypersurface, and afterwards continuing this procedure over and over. A proper solution can be obtained this way by taking the limit of these approximate solutions as $\epsilon \to 0$. An analogue of this procedure will be used in our quantum discussion.

There is also a third situation that may arise, namely constraints that are *first* class from a classical point of view but are *second* class quantum mechanically. Such constraints would arise if

$$\Delta_{ab}(p,q) = Y_{ab}{}^{c}(p,q)\, \phi_c(p,q) \ , \tag{63}$$

where, for the sake of convenience, we assume that the quantities $Y_{ab}{}^{c}(p,q)$ are all uniformly bounded away from zero and infinity, i.e., $0 < C \leq Y_{ab}{}^{c}(p,q)$ $\leq D < \infty$. In that case $\Delta_{ab}(p,q)$ would vanish on the constraint hypersurface classically. Quantum mechanically, the expression for the commutator is proportional to $\hbar$ and may be taken as

$$i[\Phi_a(P,Q),\, \Phi_b(P,Q)] = \tfrac{1}{2}[Y_{ab}{}^{c}(P,Q)\, \Phi_c(P,Q) + \Phi_c(P,Q)\, Y_{ab}{}^{c}(P,Q)] \ . \tag{64}$$

If we assume that "$\Phi_a(P,Q)|\psi\rangle_{phys} = 0$", then self-consistency requires that "$[\Phi_c(P,Q),\, Y_{ab}{}^{c}(P,Q)]|\psi\rangle_{phys} = 0$", an expression which is now proportional to $\hbar^2$. If this expression vanishes it causes no problem; if it does not vanish one says that there is a "factor ordering problem" or an "anomaly". As Jackiw has often stressed, it would be preferable to call an anomaly "quantum mechanical symmetry breaking", a phrase which more accurately describes what it is and what it does. Whatever it is called, the resultant quantum constraints are second class even though they were classically first class. As is well known, gravity falls into just this category.

In this section we take up the quantization of these more general situations involving both first and second class constraints [8].

4.2 Quantum considerations

As in previous sections, we let $\mathbb{E}$ denote the projection operator onto the quantum constraint subspace. Motivated by the classical comments given above we consider the quantity

$$\lim \langle p'', q''|\mathbb{E}\, e^{-i\epsilon\mathcal{H}}\mathbb{E}\, e^{-i\epsilon\mathcal{H}}\cdots\mathbb{E}\, e^{-i\epsilon\mathcal{H}}\mathbb{E}\,|p', q'\rangle \tag{65}$$

where the limit, as usual, is for $\epsilon \to 0$. The physics behind this expression is as follows. Reading from right to left we first impose the quantum initial value equation, and then propagate for a small amount of time (ϵ). Next we recognize that the system may have left the quantum constraint subspace, and so we project it back onto that subspace, and so on over and over. In the limit that $\epsilon \to 0$ the system remains within the quantum constraint subspace and (65) actually leads to

$$\langle p'', q''|\mathbb{E}\, e^{-iT(\mathbb{E}\mathcal{H}\mathbb{E})}\mathbb{E}\,|p', q'\rangle \;, \tag{66}$$

which clearly illustrates temporal evolution entirely within the quantum constraint subspace. If we assume that $\mathbb{E}\mathcal{H}\mathbb{E}$ is a self-adjoint operator, then we conclude that (66) describes a unitary time evolution within the quantum constraint subspace.

The expression (65) may be developed in two additional and alternative ways. First, we repeatedly insert the resolution of unity in such a way that (65) becomes

$$\lim \int \prod_{l=0}^{N} \langle p_{l+1}, q_{l+1}|\mathbb{E}\, e^{-i\epsilon\mathcal{H}}\mathbb{E}\,|p_l, q_l\rangle \prod_{l=1}^{N} d\mu(p_l, q_l) \;. \tag{67}$$

We wish to turn this expression into a formal path integral, but the procedure used previously relied on the use of unit vectors, and the vectors $\mathbb{E}\,|p, q\rangle$ are generally not unit vectors. Thus, let us rescale the factors in the integrand introducing

$$|p, q\rangle\!\rangle \equiv \mathbb{E}\,|p, q\rangle / \|\mathbb{E}\,|p, q\rangle\| \tag{68}$$

which are unit vectors. If we let $M'' \equiv \|\mathbb{E}\,|p'', q''\rangle\|$, $M' \equiv \|\mathbb{E}\,|p', q'\rangle\|$, and observe that $\|\mathbb{E}\,|p, q\rangle\|^2 = \langle p, q|\mathbb{E}\,|p, q\rangle$, it follows that (67) may be rewritten

as

$$M''M'\lim\int\prod_{l=0}^{N}\langle\!\langle p_{l+1},q_{l+1}|e^{-i\epsilon\mathcal{H}}|p_l,q_l\rangle\!\rangle\prod_{l=1}^{N}\langle p_l,q_l|\mathbb{E}\,|p_l,q_l\rangle\,d\mu(p_l,q_l)\;. \quad (69)$$

This expression is represented by the formal path integral

$$M''M'\int\exp\{i{\textstyle\int}[i\langle\!\langle p,q|(d/dt)|p,q\rangle\!\rangle-\langle\!\langle p,q|\mathcal{H}|p,q\rangle\!\rangle]\,dt\}\,\mathcal{D}_E\mu(p,q)\;, \quad (70)$$

where the new formal measure for the path integral is defined in an evident fashion from its lattice prescription. We can also reexpress this formal path integral in terms of the original bra and ket vectors in the form

$$\begin{aligned}M''M'\int\exp\{i{\textstyle\int}[i\langle p,q|\mathbb{E}\,(d/dt)\mathbb{E}\,|p,q\rangle/\langle p,q|\mathbb{E}\,|p,q\rangle\\ -\langle p,q|\mathbb{E}\,\mathcal{H}\mathbb{E}\,|p,q\rangle/\langle p,q|\mathbb{E}\,|p,q\rangle]\,dt\}\,\mathcal{D}_E\mu(p,q)\;. \quad (71)\end{aligned}$$

This last relation concludes our second route of calculation beginning with (65).

The third relation we wish to derive uses an integral representation for the projection operator $\mathbb{E}$ generally given by

$$\mathbb{E}\,={\textstyle\int}e^{-i\xi^a\Phi_a(P,Q)}\,f(\xi)\,\delta\xi \quad (72)$$

for a suitable function f. Thus we rewrite (65) in the form

$$\begin{aligned}\lim\int\langle p'',q''|e^{-i\epsilon\lambda_N^a\Phi_a}e^{-i\epsilon\mathcal{H}}e^{-i\epsilon\lambda_{N-1}^a\Phi_a}e^{-i\epsilon\mathcal{H}}\cdots e^{-i\epsilon\lambda_1^a\Phi_a}e^{-i\epsilon\mathcal{H}}e^{-i\epsilon\lambda_0^a\Phi_a}|p',q'\rangle\\ \times f(\epsilon\lambda_N)\cdots f(\epsilon\lambda_0)\,\delta\epsilon\lambda_N\cdots\delta\epsilon\lambda_0\;. \quad (73)\end{aligned}$$

Next we insert the coherent-state resolution of unity at appropriate places to find that (73) may also be given by

$$\begin{aligned}\lim\int\langle p_{N+1},q_{N+1}|e^{-i\epsilon\lambda_N^a\Phi_a}|p_N,q_N\rangle\prod_{l=0}^{N-1}\langle p_{l+1},q_{l+1}|e^{-i\epsilon\mathcal{H}}e^{-i\epsilon\lambda_l^a\Phi_a}|p_l,q_l\rangle\\ \times[\prod_{l=1}^{N}d\mu(p_l,q_l)\,f(\epsilon\lambda_l)\,\delta\epsilon\lambda_l]\,f(\epsilon\lambda_0)\,\delta\epsilon\lambda_0\;. \quad (74)\end{aligned}$$

Following the normal pattern, this last expression may readily be turned into a formal coherent-state path integral given by

$$\int \exp\{i{\textstyle\int}[p_j\dot{q}^j - H(p,q) - \lambda^a(t)\phi_a(p,q)]\,dt\}\,\mathcal{D}\mu(p,q)\mathcal{D}E(\lambda)\;, \qquad (75)$$

where $E(\lambda)$ is a measure designed so as to insert the projection operator $\mathbb{E}$ at every time slice. This usage of the Lagrange multipliers to ensure that the quantum system remains within the quantum constraint subspace is similar to their usage in the classical theory to ensure that the system remains on the classical constraint hypersurface. On the other hand, it is also possible to use the measure $E(\lambda)$ in the case of closed *first*-class constraints as well; this would be just one of the acceptable choices for the measure $C(\lambda)$ designed to put at least one projection operator $\mathbb{E}$ into the propagator.

In summary, we have established the equality of the three expressions

$$\begin{aligned}
&\langle p'',q''|\mathbb{E}\,e^{-iT(\mathbb{E}\mathcal{H}\mathbb{E})}\mathbb{E}\,|p',q'\rangle \\
&\quad = M''M'\int \exp\{i{\textstyle\int}[i\langle p,q|\mathbb{E}\,(d/dt)\mathbb{E}\,|p,q\rangle/\langle p,q|\mathbb{E}\,|p,q\rangle \\
&\qquad\qquad -\langle p,q|\mathbb{E}\,\mathcal{H}\mathbb{E}\,|p,q\rangle/\langle p,q|\mathbb{E}\,|p,q\rangle]\,dt\}\,\mathcal{D}_E\mu(p,q) \\
&\quad = \int \exp\{i{\textstyle\int}[p_j\dot{q}^j - H(p,q) - \lambda^a(t)\phi_a(p,q)]\,dt\}\,\mathcal{D}\mu(p,q)\mathcal{D}E(\lambda)\;. \qquad (76)
\end{aligned}$$

This concludes our initial derivation of path integral formulas for general constraints. Observe that we have not introduced any δ-functionals, nor, in the middle expression, reduced the number of integration variables or the limits of integration in any way even though in that expression the integral over the Lagrange multipliers has been carried out.

4.3 Universal procedure to generate single regularized constraints

The preceding section developed a functional integral approach suitable for a general set of constraints, but it had one weak point, namely, it required prior knowledge of the constraints themselves in order to choose $f(\xi)$ in (72) so as to construct the appropriate projection operator. Is there any way to construct $\mathbb{E}$ *without* prior knowledge of the form the constraints will take? The answer is *yes!*

We first observe that the evolution operator appearing in (35) may be written in the form of a lattice limit given by

$$\lim_{\epsilon\longrightarrow 0} \overleftarrow{\prod_{1\le n\le N}} \left[\mathsf{T}e^{-i\int_{(n-1)\epsilon}^{n\epsilon}\mathcal{H}(t)\,dt}\right]\left[\mathsf{T}e^{-i\int_{(n-1)\epsilon}^{n\epsilon}\lambda^a(t)\Phi_a\,dt}\right] , \tag{77}$$

where $\epsilon\equiv T/N$ and the directed product (symbol $\longleftarrow$) also respects the time ordering. Thus, this expression is simply an alternating sequence of short-time evolutions, first by $\lambda^a(t)\Phi_a$, second by $\mathcal{H}(t)$, a pattern which is then repeated $N-1$ more times. The validity of this Trotter-product form follows whenever $\mathcal{H}(t)^2+\Phi_a\delta^{ab}\Phi_b$ is essentially self adjoint for all t, $0\le t\le T$. As a slight generalization, we shall assume that $\mathcal{H}(t)^2+\Phi_aM^{ab}\Phi_b$ is essentially self adjoint for all t, $0\le t\le T$. Here the real, symmetric coefficients M^{ab} $(=M^{ba})$ are the elements of a positive-definite matrix, i.e., $\{M^{ab}\}>0$. For a finite number of constraints, $A<\infty$, it is sufficient to assume that $M^{ab}=\delta^{ab}$. Other choices for M^{ab} may be relevant when $A=\infty$. (We do not explicitly consider the case $A=\infty$ in this article; for some examples see [19].)

With all this in mind, we shall explain the construction of a formal integration procedure [20] whereby

$$\int \mathsf{T}e^{-i\int_{(n-1)\epsilon}^{n\epsilon}\lambda^a(t)\Phi_a\,dt}\,\mathcal{D}R(\lambda)=\mathbb{E}(\Phi_aM^{ab}\Phi_b\le\delta(\hbar)^2) , \tag{78}$$

and for which the integral represented by $\int\cdots\mathcal{D}R(\lambda)$ is independent of the set of operators $\{\Phi_a\}$ and the Hamiltonian operator $\mathcal{H}(t)$ for all t. First, introduce a formal Gaussian measure $\mathcal{D}S_{\gamma_n}(\lambda)$ such that

$$\begin{aligned}\int \mathsf{T}e^{-i\int_{(n-1)\epsilon}^{n\epsilon}\lambda^a(t)\Phi_a\,dt}\,\mathcal{D}S_{\gamma_n}(\lambda)&\\ =\mathcal{N}\int \mathsf{T}e^{-i\int_{(n-1)\epsilon}^{n\epsilon}\lambda^a(t)\Phi_a\,dt}\,e^{(i/4\gamma_n)\int_{(n-1)\epsilon}^{n\epsilon}\lambda^a(t)(M^{-1})_{ab}\lambda^b(t)\,dt}\,\Pi_a\mathcal{D}\lambda^a&\\ =e^{-i\epsilon\gamma_n(\Phi_aM^{ab}\Phi_b)} .&\end{aligned} \tag{79}$$

The second and last step in the construction involves an integration over γ_n given by

$$\int e^{-i\epsilon\gamma_n(\Phi_aM^{ab}\Phi_b)}\,d\Gamma(\gamma_n)$$

$$\begin{aligned}&\equiv \lim_{\zeta\to 0^+}\lim_{L\to\infty}\int_{-L}^{L} e^{-i\epsilon\gamma_n(\Phi_a M^{ab}\Phi_b)}\,\frac{\sin[\epsilon(\delta^2+\zeta)\gamma_n]}{\pi\gamma_n}\,d\gamma_n\\ &= \mathbb{E}(\epsilon\,\Phi_a M^{ab}\Phi_b \le \epsilon\,\delta^2)\\ &= \mathbb{E}(\Phi_a M^{ab}\Phi_b \le \delta^2)\ ,\end{aligned} \tag{80}$$

which achieves our goal. We note that if the final limit is replaced by $\lim_{\zeta\to 0^-}$, the result becomes $\mathbb{E}(\Phi_\alpha M^{\alpha\beta}\Phi_\beta < \delta^2)$. We normally symbolize the pair of operations by $\int\cdots\mathcal{D}R(\lambda)$, leaving the integral over γ_n implicit.

Remark: For notational simplicity throughout this article, we generally let

$$\begin{aligned}&\int e^{-i\gamma X^2}\frac{\sin(\delta^2\gamma)}{\pi\gamma}\,d\gamma\\ &\qquad\equiv \lim_{\zeta\to 0^+}\lim_{L\to\infty}\int_{-L}^{L} e^{-i\gamma X^2}\,\frac{\sin[(\delta^2+\zeta)\gamma]}{\pi\gamma}\,d\gamma\\ &\qquad= \mathbb{E}(X^2\le\delta^2)\ .\end{aligned} \tag{81}$$

With (80) we have found a single, universal procedure to create the regularized projection operator $\mathbb{E}$ from the set of constraint operators in a manner that is *completely independent of the nature of the constraints themselves.*

4.4 Basic second-class constraint example

Consider the two degree of freedom system determined by

$$I = \textstyle\int [p\dot q + r\dot s - H(p,q,r,s) - \lambda_1 r - \lambda_2 s]\,dt\ , \tag{82}$$

where we have called the variables of the second degree of freedom r, s, and H is not specified further. The coherent states satisfy $|p,q,r,s\rangle = |p,q\rangle\otimes|r,s\rangle$, which will be useful. We adopt (71) as our formal path integral in the present case, and choose [12]

$$\begin{aligned}\mathbb{E} &= \textstyle\int e^{-i(\xi_1 R+\xi_2 S)}\,e^{-(\xi_1^2+\xi_2^2)/4}\,d\xi_1 d\xi_2/(2\pi)\\ &= \mathbb{E}(R^2+S^2\le\hbar)\equiv|0_2\rangle\langle 0_2|\end{aligned} \tag{83}$$

which is a projection operator onto the fiducial vector for the second (constrained) degree of freedom only. With this choice it follows that

$$i\langle p,q,r,s|\mathbb{E}\,(d/dt)\mathbb{E}\,|p,q,r,s\rangle/\langle p,q,r,s|\mathbb{E}\,|p,q,r,s\rangle$$

$$= i\langle p,q|(d/dt)|p,q\rangle - \Im(d/dt)\ln[\langle 0_2|r,s\rangle]$$
$$= p\dot{q} - \Im(d/dt)\ln[\langle 0_2|r,s\rangle]\ , \tag{84}$$

and

$$\langle p,q,r,s|\mathbb{E}\,\mathcal{H}(P,Q,R,S)\mathbb{E}\,|p,q,r,s\rangle/\langle p,q,r,s|\mathbb{E}\,|p,q,r,s\rangle$$
$$= \langle p,q,0,0|\mathcal{H}(P,Q,R,S)|p,q,0,0\rangle$$
$$= H(p,q,0,0)\ . \tag{85}$$

Consequently, for this example, (71) becomes

$$\mathcal{M}\int \exp\{i{\textstyle\int}[p\dot{q} - H(p,q,0,0)]\,dt\}\,\mathcal{D}p\,\mathcal{D}q\ \times\ \langle r'',s''|0_2\rangle\langle 0_2|r',s'\rangle\ , \tag{86}$$

where we have used the fact that at every time slice

$$\textstyle\int\langle r,s|\mathbb{E}\,|r,s\rangle\,dr\,ds/(2\pi) = \int|\langle 0_2|r,s\rangle|^2\,dr\,ds/(2\pi) = 1\ . \tag{87}$$

Observe, in this path integral quantization, that no variables have been eliminated nor has any domain of integration been reduced; moreover, the operators R and S have remained unchanged. Also observe that the result in (86) is clearly a product of two distinct factors. The first factor describes the true dynamics as if we had solved for the classical constraints and substituted $r = 0$ and $s = 0$ in the classical action from the very beginning, while the second factor characterizes a one-dimensional Hilbert space for the second degree of freedom. Thus we can also drop the second factor completely as well as all the integrations over r and s and still retain the same physics. In this manner we recover the standard result without the use of Dirac brackets or having to initially eliminate the second-class constraints from the theory.

4.5 Conversion method

One common method to treat second-class constraints is to convert them to first-class constraints and to follow the available procedures for such systems; see, e.g., [18]. Let us first argue classically, and take as an example a single degree of freedom with canonical variables p and q, a vanishing Hamiltonian, and the second-class constraints $p = 0$ and $q = 0$. This situation may be described by the classical action

$$I = \textstyle\int[p\dot{q} - \lambda p - \xi q]\,dt\ , \tag{88}$$

where λ and ξ denote Lagrange multipliers. Next, let us introduce a second canonical pair, say r and s, and adopt the classical action

$$I' = \int [p\dot{q} + r\dot{s} - \lambda(p+r) - \xi(q-s)]\,dt\;. \tag{89}$$

Now the two constraints read $p+r=0$ and $q-s=0$ with a Poisson bracket $\{p+r, q-s\}=0$, characteristic of first-class constraints. We obtain the original problem by imposing the (consistent) *gauge conditions* that $r=0$ and $s=0$. Let us look at this example from the projection operator, coherent state approach.

In the first version with one pair of variables, we are led to the reproducing kernel

$$\begin{aligned}\langle p'', q''|\mathbb{E}(P^2+Q^2\le\hbar)|p', q'\rangle&\\ &= \langle p'', q''|0\rangle\langle 0|p', q'\rangle\\ &= e^{-\frac{1}{4}(p''^2+q''^2-2ip''q'')}\,e^{-\frac{1}{4}(p'^2+q'^2+2ip'q')}\;,\end{aligned} \tag{90}$$

which provides a “bench mark” for this example. As expected the result is a one-dimensional Hilbert space.

In the second version of this problem, we start with the expression

$$\langle p'', q'', r'', s''|\mathbb{E}((P+R)^2+(Q-S)^2\le\delta^2)|p', q', r', s'\rangle \tag{91}$$

which involves a constraint with zero in the continuous spectrum. Therefore, following previous examples, we multiply this expression with a suitable factor c_δ and take the limit as $\delta\to 0$. This factor can be chosen so that

$$\begin{aligned}\lim_{\delta\to 0} c_\delta\,\langle p'', q'', r'', s''|\mathbb{E}((P+R)^2+(Q-S)^2\le\delta^2)|p', q', r', s'\rangle&\\ = e^{-\frac{1}{4}[(p''+r'')^2+(q''-s'')^2]+\frac{1}{2}i(p''-r'')(q''-s'')}&\\ \times e^{-\frac{1}{2}i(p'-r')(q'-s')-\frac{1}{4}[(p'+r')^2+(q'-s')^2]}&\;,\end{aligned} \tag{92}$$

an expression which also describes a one-dimensional Hilbert space. This is a different (but equivalent) representation for the one-dimensional Hilbert space than the one found above. Since it is only one-dimensional we can reduce this reproducing kernel even further, in the fashion illustrated earlier, by choosing a “gauge” where $r''=s''=r'=s'=0$. When this is done the result becomes

$$e^{-\frac{1}{4}(p''^2+q''^2-2ip''q'')}\,e^{-\frac{1}{4}(p'^2+q'^2+2ip'q')}\;, \tag{93}$$

which is *identical* to the expression (90) found by quantization of the second-class constraints directly. In this manner we see how the conversion method, in which second-class constraints are turned into first-class constraints by the introduction of auxiliary degrees of freedom, appears within the projection operator, coherent state approach as well. Applications of the conversion method made within the projection operator approach may be found in [21].

4.6 Equivalent representations

In dealing with quantum mechanics, one may employ many different—yet equivalent—representations of the vectors and operators involved. While, in certain circumstances, some representations may be more convenient than others, the notion that some representations are "better" than others should be resisted.

In the context of coherent-state representations, for example, a change of the fiducial vector leads to an equivalent representation. If, for a rather general (normalized) fiducial vector $|\eta\rangle$, we set

$$|p, q; \eta\rangle \equiv e^{-iqP}\, e^{ipQ}\, |\eta\rangle\ , \tag{94}$$

then

$$\psi(p, q; \eta) \equiv \langle p, q; \eta|\psi\rangle \tag{95}$$

defines η-dependent representatives of the abstract vector $|\psi\rangle$. However, all representation-dependent aspects disappear when physical questions are asked such as

$$\textstyle\int |\psi(p, q; \eta)|^2\, (dp\, dq/2\pi) = \langle\psi|\psi\rangle\ . \tag{96}$$

More general representation issues may be addressed by using arbitrary unitary operators, say V. Thus if $|p, q\rangle$ denotes elements of one (say) coherent state basis, then $|p, q; V\rangle \equiv V^\dagger|p, q\rangle$ denotes the elements of another basis. Vector and operator representatives, $\psi(p, q; V) \equiv \langle p, q; V|\psi\rangle$ and $A(p', q'; V\ :\ p, q; V) \equiv \langle p', q'; V|\mathcal{A}|p, q; V\rangle$, respectively, provide equivalent sets of functional representatives for different V. Evidently the *physics* is unchanged in this transformation; only the intermediate *mathematical representatives* are affected. This formulation is similar to passive coordinate

transformations in other disciplines. Another version similar to active coordinate transformations is also possible. In this version the basis vectors, say $|p,q\rangle$, for all relevant (p,q), remain unchanged; instead, the abstract vectors $|\psi\rangle$ and operators $\mathcal{A}$, etc., are transformed: $|\psi\rangle \to V|\psi\rangle$, $\mathcal{A} \to V\mathcal{A}V^\dagger$, etc. It is this form of equivalence that we turn to next.

4.7 Equivalence of criteria for second-class constraints

Let us return to the simple example of second-class constraints discussed above where, classically, $p = q = 0$. In the associated quantum theory, we chose to express these constraints with the help of the projection operator $\mathbb{E} = \mathbb{E}(P^2 + Q^2 \le \hbar) = |0\rangle\langle 0|$, namely, the projection operator onto the ground state of the "Hamiltonian" $P^2 + Q^2$. In turn, this expression led directly to the coherent-state representation of $\mathbb{E}$ given by $\langle p',q'|\mathbb{E}|p,q\rangle = \langle p',q'|0\rangle\langle 0|p,q\rangle$. However, the question arises, what is special about the combination $P^2 + Q^2$? As we shall now argue, any other possible choice leads to an equivalent representation.

As a first example, consider

$$\mathbb{E}(P^2 + \omega^2 Q^2 \le \omega\hbar) = |0;\omega\rangle\langle 0;\omega| = V_\omega^\dagger\,|0\rangle\langle 0|\,V_\omega\,, \tag{97}$$

where V_ω denotes a suitable unitary operator, which establishes the equivalence for any ω, $0 < \omega < \infty$. We emphasize that we do *not* assert the unitary equivalence of $P^2 + Q^2$ and $P^2 + \omega^2 Q^2$ for any value of $\omega \ne 1$, only that $|0;\omega\rangle$ and $|0\rangle$ are unitarily related—as are *any* two unit vectors in Hilbert space.

Furthermore, there is nothing sacred about the quadratic combination. For example, for any $0 < \lambda < \infty$, consider $\mathbb{E}(P^2 + \lambda Q^4 \le \delta(\hbar)^2) \equiv |0,\lambda\rangle\langle 0,\lambda|$, where we have adjusted $\delta(\hbar)$ to the lowest eigenvalue so as to include only a single eigenvector, $|0,\lambda\rangle$. Since there exists a unitary operator V_λ such that $\langle 0,\lambda| = \langle 0|V_\lambda$, this choice of projection operator leads to an equivalent coherent-state representation as well.

More generally, we are led to reconsider the projection operator

$$\mathbb{E}(\Sigma_a \Phi_a^2 \le \delta(\hbar)^2) = \sum_{j=1}^{J} |j\rangle\langle j|\,, \tag{98}$$

where $\langle j|k\rangle = \delta_{jk}$ and $1 \leq J \leq \infty$, as determined by the choice of $\delta(\hbar)$. Since all J-dimensional subspaces are unitarily equivalent to each other (with suitable care taken when $J = \infty$), the given prescription is entirely equivalent to any other version, such as

$$\mathbb{E}(\mathcal{F}(\Phi_a) \leq \tilde{\delta}(\hbar)^2) = \sum_{\mathsf{j}=1}^{\mathsf{J}} |\mathsf{j}\rangle\langle\mathsf{j}| \;, \tag{99}$$

where $\langle \mathsf{j}|\mathsf{k}\rangle = \delta_{\mathsf{jk}}$, provided that $\tilde{\delta}(\hbar)$ may be—and is—chosen so that $\mathsf{J} = J$. Here $\mathcal{F}(\Phi_a)$ denotes a nonnegative self-adjoint operator that includes all the constraint operators, and for very small $\tilde{\delta}(\hbar)^2$ forces the spectral contribution of each constraint operator to be correspondingly small, just as is the case in (98).

In summary, the general, quadratic criterion we have adopted in (98) has been chosen for simplicity and convenience; any other restriction on the constraint operators leads to an equivalent theory, as in (99), provided that the dimensionality of $\mathbb{E}$ remains the same.

5 SELECTED EXAMPLES OF FIRST-CLASS CONSTRAINTS

5.1 General configuration space geometry

Although we shall discuss constraints that lead to a general configuration space geometry in this section, we shall for the most part use rather simple illustrative examples. To begin with let us consider the constraint

$$\sum_{j=1}^{J} (q^j)^2 = 1 \;, \tag{100}$$

a condition which puts the classical problem on a (hyper)sphere of unit radius. For convenience in what follows we shall focus as well on the case of a vanishing Hamiltonian so as to isolate clearly the consequences of the constraint independently of any dynamical effects. Adopting a standard vector inner product notation and a different kinematic term, consider the formal

path integral

$$\mathcal{M}\int \exp\{i{\textstyle\int}[-q\cdot\dot{p}-\lambda(q^2-1)]\,dt\}\,\mathcal{D}p\,\mathcal{D}q\,\mathcal{D}C(\lambda)\;, \tag{101}$$

the result of which is given by

$$\langle p'',q''|\mathbb{E}\,|p',q'\rangle \tag{102}$$

where

$$\mathbb{E}=\int_{-\infty}^{\infty} e^{-i\lambda(Q^2-1)}\,\frac{\sin(\delta\lambda)}{\pi\lambda}\,d\lambda=\mathbb{E}\,(-\delta\le Q^2-1\le\delta)\;. \tag{103}$$

In order, ultimately, to obtain a suitable reduction of the reproducing kernel in the present case, we allow for fiducial vectors other than harmonic oscillator ground states. Thus we let $|\eta\rangle$ denote a general unit vector for the moment; its required properties will emerge from our analysis. In accordance with (101), we choose a phase convention for the coherent states—in particular, in (18) we multiply by $e^{ip\cdot q}$—so that now the Schrödinger representation of the coherent states reads

$$\langle x|p,q\rangle = e^{ip\cdot x}\,\eta(x-q)\;, \tag{104}$$

which leads immediately to the expression

$$\langle p'',q''|p',q'\rangle=\int \eta^*(x-q'')\,e^{-i(p''-p')\cdot x}\,\eta(x-q')\,d^Jx\;. \tag{105}$$

Consequently, the reproducing kernel that incorporates the projection operator is given, for $0<\delta<1$, by

$$\langle p'',q''|\mathbb{E}\,|p',q'\rangle=\int_{1-\delta\le x^2\le 1+\delta}\eta^*(x-q'')\,e^{-i(p''-p')\cdot x}\,\eta(x-q')\,d^Jx\;. \tag{106}$$

Since $\mathbb{E}$ represents a projection operator, it is evident that this expression defines a reproducing kernel which admits a local integral for its inner product (for any normalized η) with a measure $d^Jp\,d^Jq/(2\pi)^J$ and an integration domain $\mathbb{R}^{2J}$.

However, if we are willing to restrict our choice of fiducial vector, we can reduce the number of integration variables and change the domain of integration in a meaningful way. Recall that the group $\mathrm{E}(J)$, the Euclidean group

in J-dimensions, consists of rotations that preserve the unit (hyper)sphere in J-dimensions, as well as J translations. As emphasized by Isham [22], this is the natural canonical group for a system confined to the surface of a (hyper)sphere in J dimensions. We can adapt our present coherent states to be coherent states for the group $\mathrm{E}(J)$ without difficulty.

To that end consider the reduction of the reproducing kernel (106) to one for which $q''^2 = q'^2 \equiv 1$. To illustrate the process as clearly as possible let us choose $J = 2$. As a consequence we introduce

$$\langle a'', b'', c''|a', b', c'\rangle \equiv \langle p'', q''|\mathbb{E}\,|p', q'\rangle_{q''^2=q'^2=1}\ , \tag{107}$$

where $a \equiv p_1$, $b \equiv p_2$, and c arises from the identification $q^1 \equiv \cos(c)$ and $q^2 \equiv \sin(c)$, all relations holding for both end points. Expressed in terms of polar coordinates, r, ϕ, the reduced reproducing kernel becomes

$$\begin{aligned}&\langle a'', b'', c''|a', b', c'\rangle\\ &\quad= \int_{|r^2-1|\le\delta} \eta^*(r, \phi - c'')\, e^{-i(a''-a')r\cos\phi - i(b''-b')r\sin\phi}\, \eta(r, \phi - c')\, r\, dr\, d\phi\,.\end{aligned} \tag{108}$$

We next seek to choose η, if at all possible, in such a way that the inner product of this new (reduced) reproducing kernel admits a local integral for its inner product. As a starting point we choose the left-invariant group measure for $\mathrm{E}(2)$ which is given by $M\, da\, db\, dc$, M a constant, with an integration domain $\mathbb{R}^2 \times S^1$. Therefore, we are led to propose that

$$\begin{aligned}&\int\int_{|r^2-1|<\delta} \eta^*(r, \phi - c'')\, e^{-i(a''-a)r\cos\phi - i(b''-b)r\sin\phi}\, \eta(r, \phi - c)\, r\, dr\, d\phi\\ &\qquad\times\int_{|\rho^2-1|<\delta} \eta^*(\rho, \theta - c)\, e^{-i(a-a')\rho\cos\theta - i(b-b')\rho\sin\theta}\, \eta(\rho, \theta - c')\, \rho\, d\rho\, d\theta\\ &\qquad\times M\, da\, db\, dc\\ &\quad= (2\pi)^2 M \int \eta^*(r, \phi - c'') e^{-i(a''-a')r\cos\phi - i(b''-b')r\sin\phi}\, \eta(r, \phi - c')\, r\, dr\, d\phi\\ &\qquad\times\int |\eta(r, c)|^2\, dc\ ,\end{aligned} \tag{109}$$

which leads to the desired result provided (i)

$$\int_0^{2\pi} |\eta(r, c)|^2\, dc = P\ , \qquad 0 < P < \infty\ , \tag{110}$$

is *independent* of r, $|r^2-1|<\delta$, and (ii) $M=[(2\pi)^2\,P]^{-1}$. Given a general nonvanishing vector $\xi(r,\phi)$, a vector satisfying (110) may always be given by

$$\eta(r,\phi)=\xi(r,\phi)/\sqrt{\textstyle\int_0^{2\pi}|\xi(r,\theta)|^2\,d\theta} \tag{111}$$

provided the denominator is positive, and which specifically leads to $P=1$. In this way we have reproduced the E(2)-coherent states of Ref. [23], even including the necessity for a small interval of integration in r, and where fiducial vectors satisfying (110) were called "surface constant".

Dynamics consistent with the constraint $q^2=1$ is obtained in the E(2) case by choosing a Hamiltonian that is a function of the coordinates on the circle, namely $\cos(\theta)$ and $\sin(\theta)$, as well as the rotation generator of E(2), i.e., $-i\partial/\partial\theta$. We refer the reader to [23] for a further discussion of E(2)-coherent states as well as a discussion of the introduction of compatible dynamics. An analogous discussion can be given for the classical constraint $q^2=1$ for any value of $J>2$.

Not only can compact (hyper)spherical configuration spaces be treated in this way, but one may also treat noncompact (hyper)pseudospherical spaces defined by the constraint

$$\Sigma_{i=1}^{I}q^{i\,2}-\Sigma_{j=I+1}^{J}q^{j\,2}=1\;, \qquad 1\le I\le J-1\;, \tag{112}$$

appropriate to the Euclidean group $\mathrm{E}(I,J-I)$. Such an analysis would lead to $\mathrm{E}(I,J-I)$-coherent states.

Finally, we comment on the constraint of a general curved configuration space which can be defined by a set of compatible constraints $\phi_a(q)=0$. Clearly these constraints satisfy $\{\phi_a(q),\phi_b(q)\}=0$, and define a $(J-A)$-dimensional configuration space in the original Euclidean configuration space $\mathrm{I\!R}^J$. The relevant projection operator $\mathrm{I\!E}=\mathrm{I\!E}\,(\Sigma\Phi_a^2(Q)\le\delta^2)$ is defined in an evident fashion, and the reproducing kernel incorporating the projection operator is defined in analogy with the prior discussion. This reproducing kernel enjoys a local integral representation for its inner product, in fact, this integral is with the same measure and integration domain as without the projection operator. What differs in the present case is that when the reproducing kernel is put on the constraint manifold, the resultant coherent states are generally *not* defined by the action of a group on a fixed fiducial vector. In short, the relevant coherent states are not group generated, which, in fact, is consistent with their most basic definition; see, e.g., [13, 24].

5.2 Finite-dimensional Hilbert space examples

Let us consider the case of two degrees of freedom with a "classical" action function given by

$$I = \int [\tfrac{1}{2}(p_1\dot{q}_1 - q_1\dot{p}_1 + p_2\dot{q}_2 - q_2\dot{p}_2) - \lambda(p_1^2 + p_2^2 + q_1^2 + q_2^2 - 4s\hbar)]\, dt \quad (113)$$

For clarity of presentation, we explicitly include $\hbar$ in our classical action, and we continue to make it explicit it throughout this section. With the present phase convention for the coherent states, the unconstrained reproducing kernel is given by

$$\begin{aligned} \langle p'', q''|p', q'\rangle &\equiv \langle z''|z'\rangle \\ &= \exp[\Sigma_{j=1}^2(-\tfrac{1}{2}|z_j''|^2 + z''^*{}_j z_j' - \tfrac{1}{2}|z_j'|^2)] \end{aligned} \quad (114)$$

where $z_j \equiv (q_j + ip_j)/\sqrt{2\hbar}$ for each of the end points.

We next observe that the constraint operator

$$\Phi =: P_1^2 + P_2^2 + Q_1^2 + Q_2^2 : -4s\hbar 1\!\!1 \quad (115)$$

has discrete eigenvalues, i.e., $2(n_1+n_2-2s)\hbar$, where n_1 and n_2 are nonnegative integers, based on the choice of $|\eta\rangle$ as the ground state for each oscillator. To satisfy $\Phi = 0$ it is necessary that $2s$ be an integer in which case the quantum constraint subspace is $(2s+1)$-dimensional. The projection operator in the present case is defined by

$$\mathbb{E} = \pi^{-1}\int_0^\pi \exp[-i\lambda(: P_1^2 + P_2^2 + Q_1^2 + Q_2^2 : -4s\hbar 1\!\!1)/\hbar]\, d\lambda \quad (116)$$

which projects onto the appropriate $(2s+1)$-dimensional subspace. It is straightforward to demonstrate that

$$\begin{aligned} \langle z''|\mathbb{E}\,|z'\rangle &= \exp[-\tfrac{1}{2}\Sigma_{j=1}^2(|z_j''|^2 + |z_j'|^2)][(2s)!]^{-1}(z''^*{}_1 z_1' + z''^*{}_2 z_2')^{2s} \\ &= \exp[-\tfrac{1}{2}\Sigma_{j=1}^2(|z_j''|^2 + |z_j'|^2)]\textstyle\sum_{k=0}^{2s}[k!(2s-k)!]^{-1}(z''^*{}_1 z_1')^k(z''^*{}_2 z_2')^{2s-k} \end{aligned} \quad (117)$$

The projected reproducing kernel in this case corresponds to a *finite* dimensional Hilbert space; nevertheless, the inner product is given by the same measure and integration domain as in the original, unprojected, infinite dimensional Hilbert space!

Of course, there are other, simpler and more familiar ways to represent a finite-dimensional Hilbert space; but any other representation is evidently equivalent to the one described here.

As the notation suggests the present quantum constraint subspace provides a natural carrier space for an irreducible representation of SU(2) with spin s. We observe that the following three expressions represent generators of the classical rotation group in their action on the constraint hypersurface:

$$\begin{aligned} s_x &= \tfrac{1}{2}(p_1p_2 + q_1q_2) \ , \\ s_y &= \tfrac{1}{2}(q_1p_2 - p_1q_2) \ , \\ s_z &= \tfrac{1}{4}(p_1^2 + q_1^2 - p_2^2 - q_2^2) \ . \end{aligned} \tag{118}$$

Thus these quantities serve as potential ingredients for a Hamiltonian which is compatible with the constraint.

Although not the subject of this section, we may also observe that an analogous discussion holds in case of the constraint

$$\phi(p,q) = p_1^2 + q_1^2 - p_2^2 - q_2^2 - 2k\hbar = 0 \ , \tag{119}$$

where k is an integer, and the resultant reduced Hilbert space is infinite dimensional for any integral k value. In this case the relevant group is SU(1,1).

5.3 Helix model

In [25], Friedberg, Lee, Pang, and Ren analyzed the so-called helix model. For details of this model (see also [26]) and its possible role as a simple analogue of the Gribov problem in non-Abelian gauge models, we refer the reader to their paper. We begin with the classical Hamiltonian for a three-degree of freedom system given by

$$H = \tfrac{1}{2}(p_1^2 + p_2^2 + p_3^2) + U(q_1^2 + q_2^2) + \lambda[g(p_2q_1 - q_2p_1) + p_3] \ , \tag{120}$$

where U denotes the potential, which hereafter, following [25], we shall choose as harmonic, namely $U(q_1^2 + q_2^2) = \omega^2(q_1^2 + q_2^2)/2$, because then this special model is fully soluble. Here, $g > 0$ is a coupling constant, and $\lambda = \lambda(t)$ is the Lagrange multiplier which enforces the single first-class constraint

$$\phi(p,q) = g(p_2q_1 - q_2p_1) + p_3 = 0 \ . \tag{121}$$

For the first two degrees of freedom we choose coherent states with the phase convention adopted for the previous example, while for the third degree of freedom we return to the original phase convention. This choice means that we consider the formal coherent state path integral given by

$$\int \exp\big(i{\textstyle\int}\{\tfrac{1}{2}(p_1\dot{q}_1 - q_1\dot{p}_1) + \tfrac{1}{2}(p_2\dot{q}_2 - q_2\dot{p}_2) + p_3\dot{q}_3$$
$$-\tfrac{1}{2}(p_1^2 + p_2^2 + p_3^2) - \tfrac{1}{2}\omega^2(q_1^2 + q_2^2)$$
$$-\lambda[g(p_2q_1 - q_2p_1) + p_3]\}\, dt\big)\, \mathcal{D}\mu(p,q)\, \mathcal{D}C(\lambda)$$
$$= \langle z_1'', z_2'', p_3'', q_3''|\, e^{-i\mathcal{H}T}\, \mathbb{E}\, |z_1', z_2', p_3', q_3'\rangle \ . \tag{122}$$

In the present case the relevant projection operator $\mathbb{E}$ is given (for $\hbar = 1$, and $0 < \delta \ll g$) by

$$\mathbb{E} = \mathbb{E}\,((gL_3 + P_3)^2 \le \delta^2) = \sum_{m=-\infty}^{\infty} \mathbb{E}\,((gm + P_3)^2 \le \delta^2)\, \mathbb{E}\,(L_3 = m) \ , \tag{123}$$

where we have used the familiar spectrum for the rotation generator L_3. If $\mathcal{H}_0$ denotes the harmonic oscillator Hamiltonian for the first two degrees of freedom, then it follows that

$$\langle z_1'', z_2'', p_3'', q_3''|\, e^{-i\mathcal{H}T}\, \mathbb{E}\, |z_1', z_2', p_3', q_3'\rangle$$
$$= \sum_{m=-\infty}^{\infty} \langle z_1'', z_2''| e^{-i\mathcal{H}_0 T} \mathbb{E}\,(L_3 = m)\, |z_1', z_2'\rangle$$
$$\times \langle p_3'', q_3''| e^{-iP_3^2 T/2} \mathbb{E}\,(-\delta \le gm + P_3 \le \delta) |p_3', q_3'\rangle$$
$$= \exp[-\tfrac{1}{2}(|z_1''|^2 + |z_2''|^2 + |z_1'|^2 + |z_2'|^2)]$$
$$\times \sum_{m=-\infty}^{\infty} \Big\{ \frac{(z_1''^* + iz_2''^*)(z_1' - iz_2')}{(z''^* - iz_2''^*)(z_1' + iz_2')} \Big\}^{m/2} I_m\big(\sqrt{(z_1''^{*2} + z_2''^{*2})(z_1'^2 + z_2'^2)}\, e^{-i\omega T}\big)$$
$$\times \exp[-\tfrac{1}{2}(gm + p_3'')^2 - \tfrac{1}{2}(gm + p_3')^2 - i\tfrac{1}{2}g^2m^2T - igm(q_3'' - q_3')]$$
$$\times \frac{2}{\sqrt{\pi}} \frac{\sin[\delta(q_3'' - q_3')]}{(q_3'' - -q_3')} + O(\delta^2) \ , \tag{124}$$

where I_m denotes the usual Bessel function.

We observe that the spectrum for the Hamiltonian agrees with the results of Ref. [25], and moreover, to leading order in δ, we have obtained gauge-invariant results, i.e., insensitivity to any choice of the Lagrange multiplier

function $\lambda(t)$, merely by projecting onto the quantum constraint subspace at $t = 0$. The constrained propagator (124) is composed with the same measure and integration domain as is the unconstrained propagator. We may also divide the constrained propagator by δ and take the limit $\delta \to 0$. The result is a new functional expression for the propagator that fully satisfies the constraint condition, but one that no longer admits an inner product with the same measure and integration domain as before.

5.4 Reparameterization invariant dynamics

Let us start with a single degree of freedom ($J = 1$) and the action

$$\textstyle\int [p\dot{q} - H(p,q)]\,dt\ . \tag{125}$$

We next promote the independent variable t to a dynamical variable, introduce s as its conjugate momentum (often called p_t), enforce the constraint $s + H(p,q) = 0$, and lastly introduce τ as a new independent variable. This modification is realized by means of the classical action

$$\textstyle\int \{pq^* + st^* - \lambda[s + H(p,q)]\}\,d\tau\ , \tag{126}$$

where $q^* = dq/d\tau$, $t^* = dt/d\tau$, and $\lambda = \lambda(\tau)$ is a Lagrange multiplier. The coherent-state path integral is constructed so that

$$\begin{aligned}\mathcal{M}\int \exp\big(i\textstyle\int\{pq^* + st^* - \lambda[s + H(p,q)]\}\,dt\big)\,\mathcal{D}p\,\mathcal{D}q\,\mathcal{D}s\,\mathcal{D}t\,\mathcal{D}C(\lambda)&\\ = \langle p'', q'', s'', t''|\mathbb{E}\,|p', q', s', t'\rangle\ ,&\end{aligned} \tag{127}$$

where

$$\begin{aligned}\mathbb{E} &= \int_{-\infty}^{\infty} e^{-i\xi[S+\mathcal{H}(P,Q)]}\,\frac{\sin(\delta\xi)}{\pi\xi}\,d\xi\\ &= \mathbb{E}\,(-\delta \le S + \mathcal{H}(P,Q) \le \delta)\ .\end{aligned} \tag{128}$$

The result in (127) and (128) represents as far as we can go without choosing $\mathcal{H}(P,Q)$.

To gain further insight into such expressions, we specialize to the case of the nonrelativistic free particle, $\mathcal{H} = P^2/2$. Then it follows that

$$\langle p'', q'', s'', t''|\mathbb{E}\,|p', q', s', t'\rangle$$

$$= \pi^{-1} \int_{-\infty}^{\infty} \exp[-\tfrac{1}{2}(k-p'')^2 - \tfrac{1}{2}(\tfrac{1}{2}k^2+s'')^2$$
$$+ik(q''-q') - i\tfrac{1}{2}k^2(t''-t')$$
$$-\tfrac{1}{2}(k-p')^2 - \tfrac{1}{2}(\tfrac{1}{2}k^2+s')^2]\, dk$$
$$\times \frac{2\sin[\delta(t''-t')]}{(t''-t')} + O(\delta^2)\;. \tag{129}$$

For any δ such that $0 < \delta \ll 1$, we observe that this expression represents a reproducing kernel which in turn defines an associated reproducing kernel Hilbert space composed, as usual, of bounded, continuous functions given, for arbitrary complex numbers $\{\alpha_k\}$, phase-space points $\{p_k, q_k, s_k, t_k\}$, and $K < \infty$, by

$$\psi(p,q,s,t) \equiv \sum_{k=0}^{K} \alpha_k \langle p,q,s,t|\mathbb{E}\,|p_k,q_k,s_k,t_k\rangle\;, \tag{130}$$

or as the limit of Cauchy sequences of such functions in the norm defined by means of the inner product given by

$$(\psi,\psi) = \textstyle\int |\psi(p,q,s,t)|^2\, dp\,dq\,ds\,dt/(2\pi)^2 \tag{131}$$

integrated over $\mathbb{R}^4$.

Let us next consider the reduction of the reproducing kernel given by

$$\langle p'',q'',t''|p',q',t'\rangle$$
$$\equiv \lim_{\delta\to 0} \frac{1}{4\sqrt{\pi}\,\delta} \int \langle p'',q'',s'',t''|\mathbb{E}\,|p',q',s',t'\rangle\, ds''\, ds'$$
$$= \pi^{-1/2} \int \exp[-\tfrac{1}{2}(k-p'')^2 - \tfrac{1}{2}(k-p')^2$$
$$+ik(q''-q') - i\tfrac{1}{2}k^2(t''-t')]\, dk\;, \tag{132}$$

which in turn generates a *new* reproducing kernel in the indicated variables. For the resultant kernel it is straightforward to demonstrate, for any t, that

$$\int \langle p'',q'',t''|p,q,t\rangle\langle p,q,t|p',q',t'\rangle\, dp\,dq/(2\pi) = \langle p'',q'',t''|p',q',t'\rangle\;. \tag{133}$$

This relation implies that the span of the vectors $\{|p,q\rangle \equiv |p,q,0\rangle\}$ is identical with the span of the vectors $\{|p,q,t\rangle\}$, meaning further that the states

$\{|p,q,t\rangle\}$ form a set of *extended coherent states*, which are "extended" with respect to t in the sense of Ref. [27]. Observe how the time variable has become distinguished by the criterion (133). Consequently, we may properly interpret

$$\langle p'',q'',t''|p',q',t'\rangle \equiv \langle p'',q''|e^{-i(P^2/2)(t''-t')}|p',q'\rangle \ , \tag{134}$$

namely, as the conventional, single degree of freedom, coherent-state matrix element of the evolution operator appropriate to the free particle.

To further demonstrate this interpretation as the dynamics of the free particle, we may pass to sharp q matrix elements with the observation that

$$\begin{aligned}
&\langle q''|e^{-i(P^2/2)(t''-t')}|q'\rangle \\
&\qquad \equiv \frac{\pi^{1/2}}{(2\pi)^2}\int \langle p'',q''|e^{-i(P^2/2)(t''-t')}|p',q'\rangle \, dp''\, dp' \\
&\qquad = \frac{1}{2\pi}\int \exp[\, ik(q''-q') - i\tfrac{1}{2}k^2(t''-t')]\, dk \\
&\qquad = \frac{e^{i(q''-q')^2/2(t''-t')}}{\sqrt{2\pi i(t''-t')}} \ ,
\end{aligned} \tag{135}$$

which is clearly the usual result.

5.5 Elevating the Lagrange multiplier to an additional dynamical variable

Sometimes it is useful to consider an alternative formulation of a system with constraints in which the initial Lagrange multipliers are regarded as dynamical variables, complete with their own conjugate variables, and to introduce new constraints as needed. For example, let us start with a single degree of freedom system with a single first-class constraint specified by the action functional

$$\textstyle\int [p\dot{q} - H(p,q) - \lambda\phi(p,q)]\, dt \ , \tag{136}$$

where $\phi(p,q)$ represents the constraint and λ the Lagrange multiplier. Instead, let us replace this action functional by

$$\textstyle\int [p\dot{q} + \pi\dot{\lambda} - H(p,q) - \sigma\pi - \theta\phi(p,q)]\, dt \ . \tag{137}$$

In this expression we have introduced π as the canonical conjugate to λ, the Lagrange multiplier σ to enforce the constraint $\pi = 0$, and the Lagrange multiplier θ to enforce the original constraint $\phi = 0$. Observe that $\{\pi, \phi(p,q)\} = 0$, and therefore the constraints remain first class in the new form. The path integral expression for the extended form reads

$$\begin{aligned}\mathcal{M}\int \exp\{i{\textstyle\int}[p\dot{q} + \pi\dot{\lambda} - H(p,q) - \sigma\pi - \theta\phi(p,q)]\,dt\}\,\mathcal{D}p\,\mathcal{D}q\,\mathcal{D}\pi\,\mathcal{D}\lambda\,\mathcal{D}C(\sigma,\theta)\\ = \langle p'', q'', \pi'', \lambda''|e^{-i\mathcal{H}T}\mathbb{E}\,|p', q', \pi', \lambda'\rangle\ . \qquad (138)\end{aligned}$$

In this expression, we may choose

$$\mathbb{E} = \mathbb{E}\,(\Phi(P,Q)^2 \le \delta^2)\,\mathbb{E}\,(\Pi^2 \le \delta'^2) \qquad (139)$$

involving two possibly distinct reglarization parameters. Consequently, the complete propagator factors into two terms,

$$\begin{aligned}&\langle p'', q'', \pi'', \lambda''|e^{-i\mathcal{H}T}\mathbb{E}\,|p', q', \pi', \lambda'\rangle\\ &\quad = \langle p'', q''|e^{-i\mathcal{H}T}\mathbb{E}\,(\Phi(P,Q)^2 \le \delta^2)|p', q'\rangle\\ &\quad \times\langle \pi'', \lambda''|\mathbb{E}\,(\Pi^2 \le \delta'^2)|\pi', \lambda'\rangle\ . \qquad (140)\end{aligned}$$

The first factor is exactly what would be found by the appropriate path integral of the original classical system with only the single constraint $\phi(p,q) = 0$ and the single Lagrange multiplier λ. The second factor represents the modification introduced by considering the extended system. Note, however, that with a suitable δ'-limit the second factor reduces to a product of terms, one depending on the " $''$ " arguments, the other depending on the " $'$ " arguments, just as was the case previously. This result for the second factor implies that it has become the reproducing kernel for a *one-dimensional* Hilbert space, and when multiplied by the first factor it may be ignored entirely. In this way it is found that the quantization of the original and extended systems leads to identical results.

6 SPECIAL APPLICATIONS

6.1 Algebraically inequivalent constraints

The following example is suggested by Problem 5.1 in Ref. [5]. Consider the two-degree of freedom system with vanishing Hamiltonian described by the

classical action

$$I = \textstyle\int (p_1\dot{q}_1 + p_2\dot{q}_2 - \lambda_1 p_1 - \lambda_2 p_2)\, dt \; . \tag{141}$$

The equations of motion become

$$\dot{q}_j = \lambda_j \; , \qquad \dot{p}_j = 0 \; , \qquad p_j = 0 \; , \qquad\quad j = 1, 2 \; . \tag{142}$$

Evidently the Poisson bracket $\{p_1, p_2\} = 0$.

As a second version of the same dynamics, consider the classical action

$$I = \textstyle\int (p_1\dot{q}_1 + p_2\dot{q}_2 - \lambda_1 p_1 - \lambda_2 e^{cq_1} p_2)\, dt \; , \tag{143}$$

which leads to the equations of motion

$$\dot{q}_1 = \lambda_1 \; , \quad \dot{q}_2 = \lambda_2 e^{cq_1} \; , \quad \dot{p}_1 = -c\lambda_2 e^{cq_1} p_2 \; , \quad \dot{p}_2 = 0 \; , \quad p_1 = e^{cq_1} p_2 = 0 \tag{144}$$

Since $e^{cq_1}p_2 = 0$ implies that $p_2 = 0$, it follows that the two formulations are equivalent despite the fact that in the second case $\{p_1, e^{cq_1}p_2\} = -c\, e^{cq_1}p_2$, which has a fundamentally different algebraic structure when $c \neq 0$ as compared to $c = 0$.

Let us discuss these two examples from the point of view of a coherent state, projection operator quantization. For the first version we consider

$$\mathcal{M} \int \exp[i \textstyle\int (p_1\dot{q}_1 + p_2\dot{q}_2 - \lambda_1 p_1 - \lambda_2 p_2)\, dt]\, \mathcal{D}p\, \mathcal{D}q\, \mathcal{D}C(\lambda) \; , \tag{145}$$

defined in a fashion to yield

$$\langle p'', q'' | \mathbb{E} | p', q' \rangle \tag{146}$$

where, for ease of evaluation, we may choose

$$\mathbb{E} = \mathbb{E}(P_1^2 \leq \delta^2)\mathbb{E}(P_2^2 \leq \delta^2) \; . \tag{147}$$

In particular this choice leads to the fact that

$$\begin{aligned} &\langle p'', q'' | \mathbb{E} | p', q' \rangle \\ &= \pi^{-1} \prod_{l=1}^{2} \int_{-\delta}^{\delta} \exp[-\tfrac{1}{2}(k_l - p_l'')^2 + ik_l(q_l'' - q_l') - \tfrac{1}{2}(k_l - p_l')^2]\, dk_l \; . \end{aligned} \tag{148}$$

Let us reduce this reproducing kernel, in particular, by multiplying this expression by $\pi/(2\delta)^2$ and passing to the limit $\delta \to 0$. The result is the reduced reproducing kernel given by

$$\exp[-\tfrac{1}{2}(p_1''^2 + p_2''^2)]\,\exp[-\tfrac{1}{2}(p_1'^2 + p_2'^2)]\;, \tag{149}$$

which clearly characterizes a particular representation of a one-dimensional Hilbert space in which every vector is proportional to $\exp[-\frac{1}{2}(p_1^2+p_2^2)]$. This example, of course, is related to the reduction examples given earlier. Moreover, we can introduce an integral representation over the remaining p variables for the inner product if we so desire.

Let us now turn attention to the second formulation of the problem by focussing [for a different $C(\lambda)$] on

$$\mathcal{M}\int \exp[i{\textstyle\int}(p_1\dot{q}_1 + p_2\dot{q}_2 - \lambda_1 p_1 - \lambda_2 e^{cq_1} p_2)\,dt]\,\mathcal{D}p\,\mathcal{D}q\,\mathcal{D}C(\lambda)\;. \tag{150}$$

This expression again leads (for a different $\mathbb{E}$) to

$$\langle p'', q''|\mathbb{E}|p', q'\rangle\;, \tag{151}$$

where in the present case the fully *reduced* form of this expression is proportional to

$$\begin{aligned}\int &\exp[-\tfrac{1}{2}(k_2 - p_2'')^2 + ik_2(q_2'' - q_2') - \tfrac{1}{2}(k_2 - p_2')^2]\\ &\quad\times\exp[-\tfrac{1}{2}(k_1 - p_1'')^2 + ik_1 q_1'' - \tfrac{1}{2}i\lambda_1 k_1]\\ &\quad\times\exp[-ixk_1 - i\lambda_2 e^{cx}k_2 + ix\kappa_1]\\ &\quad\times\exp[-\tfrac{1}{2}i\lambda_1\kappa_1 - i\kappa_1 q_1' - \tfrac{1}{2}(\kappa_1 - p_1')^2]\\ &\qquad\times dk_2\,dk_1\,dx\,d\kappa_1\,d\lambda_1\,d\lambda_2\;.\end{aligned} \tag{152}$$

When normalized appropriately, this expression is evaluated as

$$\exp[-\tfrac{1}{2}(p_1''^2 + p_2''^2 + icp_1'')]\,\exp[-\tfrac{1}{2}(p_1'^2 + p_2'^2 - icp_1')]\;, \tag{153}$$

which once again represents a one-dimensional Hilbert space although it has a different representation than in the case $c = 0$.

Thus we have obtained a c-dependent family of distinct but equivalent quantum representations for the same Hilbert space, reflecting the c-dependent family of equivalent classical solutions.

6.2 Irregular constraints

In discussing constraints one often pays considerable attention to the regularity of the expressions involved. Consider, once again, the simple example of a single constraint $p=0$ as illustrated by the classical action

$$I = \textstyle\int (p\dot{q} - \lambda p)\, dt\ . \tag{154}$$

The equations of motion read $\dot{q}=\lambda$, $\dot{p}=0$, and $p=0$. On the other hand, one may ask about imposing the constraint $p^3=0$ or possibly $p^{1/3}=0$, etc., instead of $p=0$. Let us incorporate several such examples by studying the classical action

$$\textstyle\int (p\dot{q} - \lambda p|p|^\gamma)\, dt\ , \qquad \gamma > -1\ . \tag{155}$$

Here the equations of motion include $\dot{q}=\lambda(\gamma+1)\,|p|^\gamma$ which, along with the constraint $p|p|^\gamma=0$, may cause some difficulty in seeking a classical solution of the equations of motion. When $\gamma\neq 0$, such constraints are said to be *irregular* [5]. It is clear from (9) that irregular constraints lead to considerable difficulty in conventional phase-space path integral approaches.

Let us examine the question of irregular constraints from the point of view of a coherent state, projection operator, phase-space path integral quantization. We first observe that the operator $P|P|^\gamma$ is well defined by means of its spectral decomposition. Moreover, for any $\gamma > -1$, it follows that

$$\begin{aligned}&\int e^{-i\xi P|P|^\gamma}\frac{\sin(\delta^{\gamma+1}\xi)}{\pi\xi}\,d\xi\\ &\qquad = \mathbb{E}(-\delta^{\gamma+1}\le P|P|^\gamma\le\delta^{\gamma+1})\\ &\qquad = \mathbb{E}(-\delta\le P\le\delta)\ .\end{aligned} \tag{156}$$

Thus, from the operator point of view, it is possible to consider the constraint operator $P|P|^\gamma$ just as easily as P itself. In particular, it follows that

$$\langle p'',q''|\mathbb{E}|p',q'\rangle = \mathcal{M}\int \exp[i\textstyle\int(p\dot{q}-\lambda p|p|^\gamma)\,dt]\,\mathcal{D}p\,\mathcal{D}q\,\mathcal{D}C_\gamma(\lambda)\ , \tag{157}$$

where we have appended γ to the measure for the Lagrange multiplier λ to emphasize the dependence of that measure on γ. The reduction of the reproducing kernel proceeds as with the cases discussed earlier, and we determine

for all γ that

$$\lim_{\delta\to 0}\frac{\sqrt{\pi}}{(2\delta)}\langle p'',q''|\mathbb{E}|p',q'\rangle = e^{-\frac{1}{2}(p''^2+p'^2)}\ , \tag{158}$$

representative of a one-dimensional Hilbert space. Note that, like the classical theory, the ultimate form of the quantum theory is independent of γ.

It is natural to ask how one is to understand this acceptable behavior for the quantum theory for irregular constraints and the difficulties they seem to present to the classical theory. Just like the classical and quantum Hamiltonians, the connection between the classical and quantum constraints is given by

$$\phi(p,q)\equiv\langle p,q|\Phi(P,Q)|p,q\rangle = \langle 0|\Phi(P+p,Q+q)|0\rangle\ . \tag{159}$$

With this rule we typically find that $\phi(p,q)\neq\Phi(p,q)$ due to the fact that $\hbar\neq 0$, but the difference between these expressions is generally qualitatively unimportant. In certain circumstances, however, that difference is qualitatively significant even though it is quantitatively very small. Since that difference is $O(\hbar)$, let us explicitly exhibit the appropriate $\hbar$-dependence hereafter.

First consider the case of $\gamma = 2$. In that case

$$\langle p,q|P^3|p,q\rangle = \langle 0|(P+p)^3|0\rangle = p^3+3\langle P^2\rangle p\ , \tag{160}$$

where we have introduced the shorthand $\langle(\cdot)\rangle\equiv\langle 0|(\cdot)|0\rangle$. Since $\langle P^2\rangle = \hbar/2$ it follows that for the quantum constraint P^3, the corresponding classical constraint function is given by $p^3+(3\hbar/2)p$. For $|p|\gg\sqrt{\hbar}$, this constraint is adequately given by p^3. However, when $|p|\ll\sqrt{\hbar}$—*as must eventually be the case in order to actually satisfy the classical constraint*—then the functional form of the constraint is effectively $(3\hbar/2)p$. In short, if the quantum constraint operator is P^3, then the classical constraint function is in fact *regular* when the constraint vanishes.

A similar analysis holds for a general value of γ. The classical constraint is given by

$$\begin{aligned}\phi_\gamma(p) &= (\pi\hbar)^{-1/2}\int(k+p)|k+p|^\gamma e^{-k^2/\hbar}\,dk\\ &= (\pi\hbar)^{-1/2}\int k|k|^\gamma e^{-(k-p)^2/\hbar}\,dk\ .\end{aligned} \tag{161}$$

For $|p| \gg \sqrt{\hbar}$ this expression effectively yields $\phi_\gamma(p) \simeq p|p|^\gamma$. On the other hand, for $p \approx 0$, and more especially for $|p| \ll \sqrt{\hbar}$, this expression shows that the constraint function vanishes *linearly*, specifically as $\phi_\gamma(p) \simeq \kappa\, p$, where

$$\kappa \equiv 2(\hbar^\gamma/\pi)^{1/2} \int y^2 |y|^\gamma e^{-y^2}\, dy = 2(\hbar^\gamma/\pi)^{1/2}\Gamma((\gamma+3)/2) \equiv \hbar^{\gamma/2}\kappa_o\ . \quad (162)$$

A rough, but qualitatively correct expression for this behavior is given by

$$\phi_\gamma(p) \simeq \kappa_o\, p(\hbar + p^2 \kappa_o^{-2/\gamma})^{\gamma/2}\ . \quad (163)$$

Thus, from the present point of view, irregular constraints do not arise from consistent quantum constraints; instead, irregular constraints arise as limiting expressions of consistent, regular classical constraints as $\hbar \to 0$.

7 OTHER APPLICATIONS OF THE PROJECTION OPERATOR APPROACH

There have been several cases in which the projection operator has been used to study constrained systems. Shabanov [9, 10] as well as Govaerts and Klauder [28] have applied the projection operator formalism to a simple $0+1$ model of a gauge theory. Govaerts [29] applied the projection operator scheme to study the relativistic particle in a reparameterization invariant form. Shabanov and Klauder have studied both first-class [30] and second-class constraint [21] situations from the point of view of projection operator quantization. In addition, they have discussed in a general way the application of projection operator techniques to gauge theory [31]. Fermion systems have been treated, e.g., in [32]. Shabanov has incorporated the projection operator into his Physics Reports [6] review of gauge theories, and developed an algorithm for how the projection operator approach may be incorporated into lattice gauge theory calculations. Shabanov has also shown how the projection operator approach may be especially useful in ensuring constraints are satisfied in an ion-surface interaction [33]. In addition, Klauder [19] has applied the projection operator method in a study of quantum gravity. Finally, a U(1) Chern-Simons model has been studied and solved with the projection operator method using coherent states in [34].

Projection operators have also been used previously in the study of constrained system quantization. For example, as noted earlier, some aspects of a coherent state quantization procedure that emphasized projection operators for systems with closed first-class constraints have been presented by Shabanov [9]. In addition, we thank M. Henneaux for his thoughtful comments as this approach was being developed, as well as for pointing out that projection operators for closed first-class constraints also appear in the text of Henneaux and Teitelboim [5]. Please note that this very short list does not pretend to be complete regarding prior considerations of projection operator investigations in connection with constrained systems.

Acknowledgements

The present paper represents a summary of some of the author's principal contributions to the projection operator approach for the quantization of systems with constraints for the four years from 1996 through 1999. It is a pleasure to thank my coauthors J. Govaerts and S. Shabanov who have shared in this general project. In addition to these two individuals, thanks are also extended to M. Henneaux and B. Whiting for many discussions regarding constraints and their quantization.

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